Communications Network Design

lecture 10

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Concave costs

When costs are concave, the network design problem has properties like single path routing. A common example is linear costs. Also we present a simple heursitic approach.

The lecture considers some generic issues regarding concave costs, and the resultant multi-commodity flow optimization problem, which is a general form of the network design problem.

Notation recap

Mostly as before (lecture 6)

- A **network** is a graph $G(N,E)$, with **nodes** $N = \{1,2,\ldots n\}$ and **links** $E \subseteq N \times N$
- Offered traffic between O-D pair (p,q) is t_{pq}
- The set of all **paths** in $G(N,E)$ is $P = \cup_{[p,q]\in K} P_{pq}$
- Each link $e \in E$ has
  - a **capacity**, denoted by $r_e (\geq 0)$
  - a **distance** $d_e (\geq 0)$
  - a **load** $f_e (\geq 0)$
- The vector $\mathbf{x} = (x_\mu : \mu \in P)$ is called the **routing**

$$f_e = \sum_{\mu \in P : e \in \mu} x_\mu$$

Multicommodity flow problem

Completely general case

- objective: minimize some cost function
  - construction costs based on capacities r_e
  - performance costs (e.g. delays, reliability, ...) based on r_e and f_e
- input:
  - a set of nodes N
  - forecast traffic demands t_{pq}
- constraints are flow based (as before)
  - loads on links are implied by routing of traffic
  - link loads $\leq$ capacities

Call it the **multicommodity flow problem**

A simplified problem

- in general costs depend on r_e and f_e
  - lets us start a little simpler
  - only include construction costs
    - ⋆ not performance costs
- assume we choose $r_e = f_e$
  - choose capacities to carry required loads
    - ⋆ could include some overhead, e.g. $r_e = \gamma f_e$ for some $\gamma > 1$
- problem simplifies to choosing which links we need in our network
  - it becomes an integer programming problem
  - it has a direct relationship to least-cost routing on a complete graph

Formal problem specification

Formal problem specification:

$$
\begin{array}{llrll}
\text{(P)} & \text{min.} & C(\mathbf{f}) & = \sum_{e \in E} c_e(f_e) & \\
& \text{s.t.} & f_e & = \sum_{\mu \in P : e \in \mu} x_\mu & \forall e \in E. \\
& & x_\mu & \geq 0 & \forall \mu \in P \\
& & \sum_{\mu \in P_{pq}} x_\mu & = t_{pq} & \forall [p,q] \in K.
\end{array}
$$

Where we then take $r_e = \gamma f_e, \forall e \in E$

This looks the same as for routing, but the set E is the set of **all possible** links, rather than a given set, and the cost function C will be different (though still separable).

Typical cost function

- assume the cost function is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$
- assume the cost function nondecreasing
- assume the cost function is separable

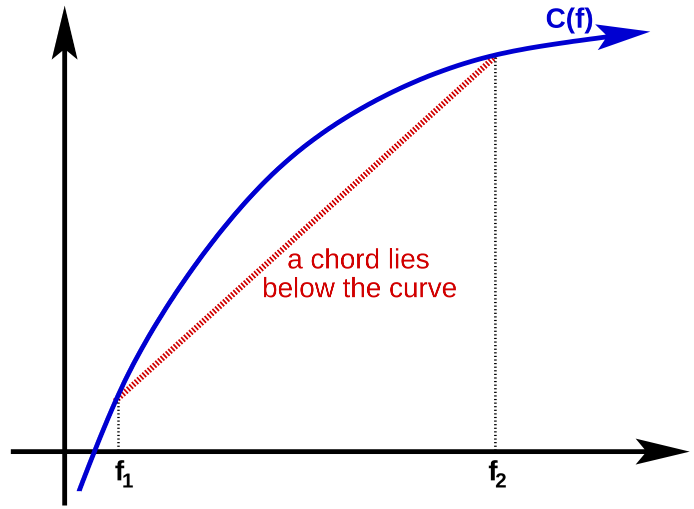
$$C(\mathbf{f}) = \sum_e c_e(f_e)$$

- assume the cost function is concave
  $C(\mathbf{f})$ is concave over Ω if for all $\lambda \in [0,1]$, and all feasible loads $\mathbf{f}_1, \mathbf{f}_2 \in \Omega$,

$$C(\lambda \mathbf{f}_1 + (1-\lambda)\mathbf{f}_2) \geq \lambda C(\mathbf{f}_1) + (1-\lambda) C(\mathbf{f}_2)$$

  - chords lie below the function

Concave



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Concave costs

- concave costs represent "economy of scales"
  - operations at a larger scale have a smaller marginal cost, e.g. $\frac{\partial c_e}{\partial f_e}$ is decreasing
  - operations at a larger scale have a smaller average cost, e.g. $\frac{c_e(f_e)}{f_e}$ is decreasing
- alternative view "multiplexing gain"
  - multiplexed (grouped) traffic has a lower relative variance, and so is less "bursty"
  - less overhead is required for smoother traffic
- Example

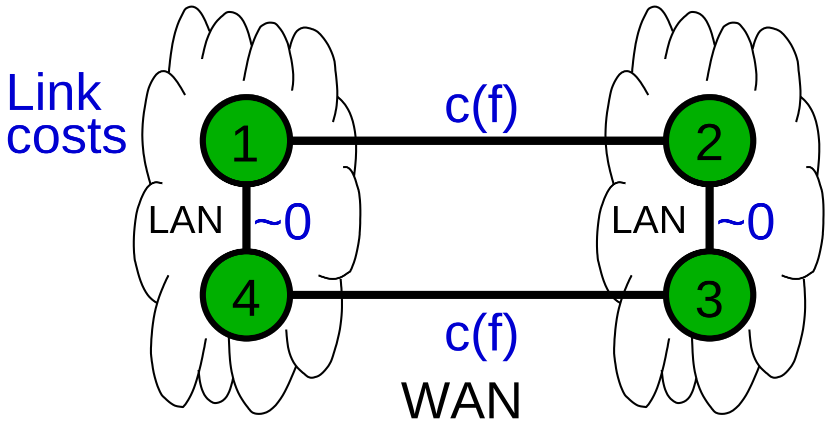
$$c_e(f_e) = k_e f_e^{\alpha}, \quad k_e = \text{constant}, \ \alpha \in (0.4, 0.6)$$

Concave costs and routing

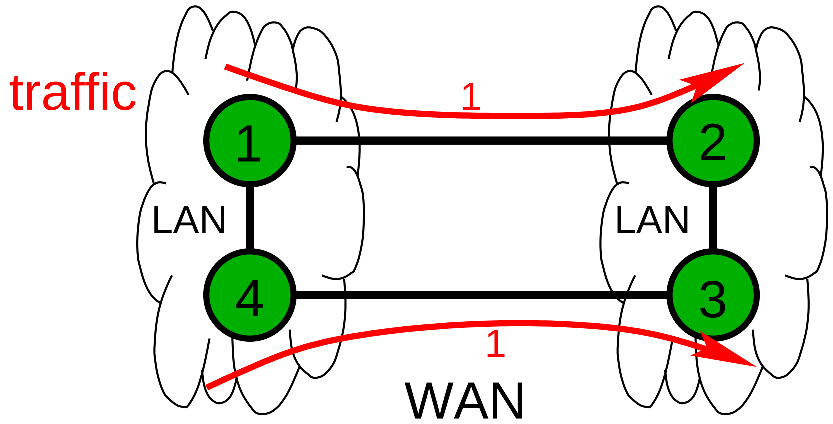
- we have so far (today) ignored routing
- nice result that shows for concave costs, we only need to consider single path routing (no load sharing)

Proposition: If $C(\mathbf{f})$ is a concave cost function of load $\mathbf{f}$, then the minimum is attained by routing t_{pq} on a single path $\hat{\mu}_{pq}$ for all O-D pairs $[p,q] \in K$.

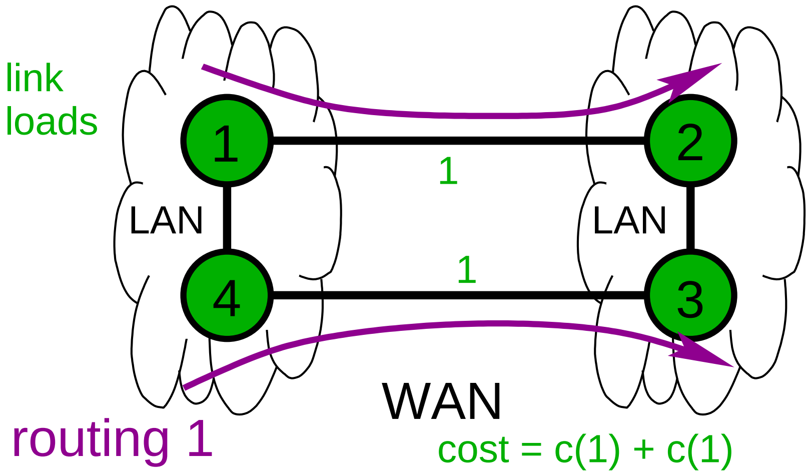
Example of single path routing



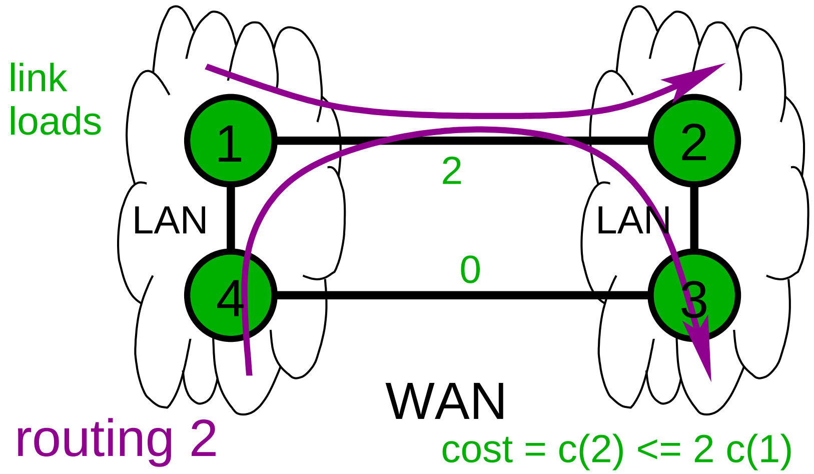
Example of single path routing



Example of single path routing



Example of single path routing



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Concave costs and routing: proof

Proof: Let us take two paths $\mu_1, \mu_2 \in P_{pq}$. Suppose there is a routing $\mathbf{x} = (x_\mu : \mu \in P)$ such that the traffic between the O-D pair $[p,q]$ is routed across both μ_1 and μ_2, i.e. $x_{\mu_1} > 0$ and $x_{\mu_2} > 0$.

Let $\mathbf{f}$ be the link loads induced by $\mathbf{x}$; so

$$f_e = \sum_{\mu: e \in \mu} x_\mu$$

Consider two cases:

- the traffic x_{μ_2} on μ_2 is moved to μ_1 inducing loads $\mathbf{f}^{(1)}$
- the traffic x_{μ_1} on μ_1 is moved to μ_2 inducing loads $\mathbf{f}^{(2)}$

Concave costs and routing: proof

The net result is:

$$\mathbf{f} = \frac{x_{\mu_1}\mathbf{f}^{(1)} + x_{\mu_2}\mathbf{f}^{(2)}}{x_{\mu_1} + x_{\mu_2}} = \frac{x_{\mu_1}}{x_{\mu_1} + x_{\mu_2}}\mathbf{f}^{(1)} + \frac{x_{\mu_2}}{x_{\mu_1} + x_{\mu_2}}\mathbf{f}^{(2)} \qquad (1)$$

and therefore, for all $e \in E$,

$$f_e = \frac{x_{\mu_1} f_e^{(1)} + x_{\mu_2} f_e^{(2)}}{x_{\mu_1} + x_{\mu_2}} \qquad (2)$$

- In both cases links $e \notin \mu_1, \mu_2$ and links $e \in \mu_1, \mu_2$ have load unaltered, e.g. $f_e^{(1)} = f_e^{(2)} = f_e$.
- Only those links on precisely one of the paths μ_1, μ_2 have loads altered by this process.

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Concave costs and routing: proof

In more detail: check this out for links $e \in E$:

- if $e \in \mu_1$ and $e \in \mu_2$ then $f_e^{(1)} = f_e^{(2)} = f_e$ so equation (2) correctly gives the load as f_e.
- if $e \notin \mu_1$ and $e \notin \mu_2$ then $f_e^{(1)} = f_e^{(2)} = f_e$, and (2) is OK.
- if $e \in \mu_1$ but $\notin \mu_2$ then $f_e^{(1)} = f_e + x_{\mu_2}$ and $f_e^{(2)} = f_e - x_{\mu_1}$. So RHS of (2) above gives
  $$\frac{x_{\mu_1} f_e^{(1)} + x_{\mu_2} f_e^{(2)}}{x_{\mu_1} + x_{\mu_2}} = \frac{x_{\mu_1}(f_e + x_{\mu_2}) + x_{\mu_2}(f_e - x_{\mu_1})}{x_{\mu_1} + x_{\mu_2}} = f_e$$
- if $e \notin \mu_1$ but $\in \mu_2$ then $f_e^{(1)} = f_e - x_{\mu_2}$ and $f_e^{(2)} = f_e + x_{\mu_1}$. So RHS of (2) above gives
  $$\frac{x_{\mu_1} f_e^{(1)} + x_{\mu_2} f_e^{(2)}}{x_{\mu_1} + x_{\mu_2}} = \frac{x_{\mu_1}(f_e - x_{\mu_2}) + x_{\mu_2}(f_e + x_{\mu_1})}{x_{\mu_1} + x_{\mu_2}} = f_e$$

Concave costs and routing: proof

Take $\lambda = \frac{x_{\mu_1}}{x_{\mu_1} + x_{\mu_2}} \in (0,1)$ and $1 - \lambda = \frac{x_{\mu_2}}{x_{\mu_1} + x_{\mu_2}} \in (0,1)$.
When C is concave. By definition, for all $\lambda \in [0,1]$,

$$C(\lambda \mathbf{f}_1 + (1-\lambda)\mathbf{f}_2) \geq \lambda C(\mathbf{f}_1) + (1-\lambda)C(\mathbf{f}_2)$$

Given that $\mathbf{f} = \lambda \mathbf{f}^{(1)} + (1-\lambda)\mathbf{f}^{(2)}$ we get

$$C(\mathbf{f}) \geq \lambda C(\mathbf{f}^{(1)}) + (1-\lambda)C(\mathbf{f}^{(2)})$$

If $C(\mathbf{f}^{(1)}) \leq C(\mathbf{f}^{(2)})$, then $\lambda C(\mathbf{f}^{(1)}) + (1-\lambda)C(\mathbf{f}^{(2)}) \geq C(\mathbf{f}^{(1)})$ and therefore, $C(\mathbf{f}) \geq C(\mathbf{f}^{(1)})$. This means the traffic can t_{pq} can all be re-routed onto μ_1 with less cost.

If $C(\mathbf{f}^{(1)}) \geq C(\mathbf{f}^{(2)})$ then, re-route traffic t_{pq} onto μ_2. □

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Concave costs and routing

- The result above means that with concave costs
  - we can assume that single paths are used for end-to-end demands.
- Heuristic for network design
  - adapt the Frank-Wolfe method
    - remember this was used for routing with **convex** costs
  - assumptions
    - we start with a single path routing $\mathbf{x}$
    - the corresponding induced load is $\mathbf{f}$
    - the routing is **not** a shortest path routing

Heuristic Method

- If all traffic is allocated to a shortest path, STOP.
- Else, select for all $k \in K$, a shortest length path $\hat{\mu}_k$ of length $l_{\hat{\mu}_k}$.
- Allocate t_k to its shortest path $\hat{\mu}_k$ for all $k \in K$.
- Call this routing $\mathbf{z}$.
- Re-calculate shortest paths; go to first step.

Note we have concave cost, so there is no guarantee that the shortest path routing we find will be the minimal cost routing!

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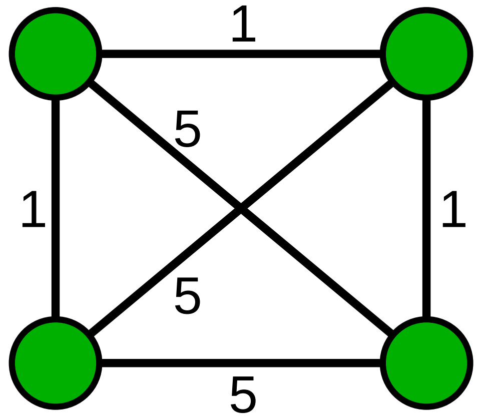
The point

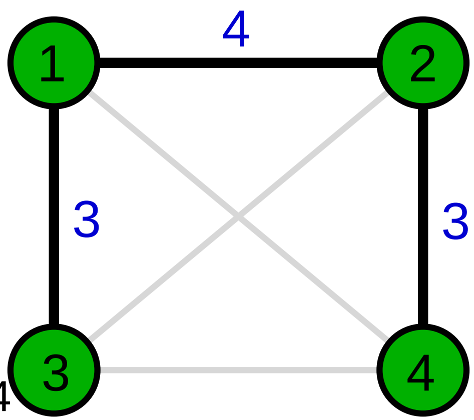
- Routing and capacity are intricately linked
- We can solve the capacity problem (for the cases above) by solving the routing problem on a **complete graph**.
- Any link with zero traffic is eliminated
- other links have capacities designed to carry traffic plus some overhead.
- Different types of cost
  - routing ⇒ convex costs ⇒ SPF
  - construction ⇒ concave costs ⇒ unique routing
- special case: **linear costs**
  - best of both cases: unique SPF routing

Example with linear costs



Example with linear costs

O–D	paths
1–2	1–2
1–3	1–3
1–4	1–2–4
2–3	2–1–3
2–4	2–4
3–4	3–1–2–4



References

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