Information Theory and Networks

Lecture 16: Gambling and Information Theory

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Part I

Gambling and Information Theory

If fighting is sure to result in victory, then you must fight, even though the ruler forbid it;
If fighting will not result in victory, then you must not fight even at the ruler's bidding.

Sun Tzu, The Art of War, Chapter 10, 23

Section 1

Horse Racing

Fixed-Odds Horse Racing

- Pool of money betting on horses
  - odds: expressed as o-for-1 or $(o-1)$-to-1
  - probability of success by probability of failure
  - assume no track take, no commissions
- What's the best strategy?
  - one-off bet
  - multiple ongoing bets, or *parlayed bets*

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Example

- Here, only bet on horse win (not other bets like place etc.)
- Odds are fixed by a bookie
- We use o-for-1 convention

Horse	Odds
1	10
2	2
3	20
4	5

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Betting Strategies

- One-off bet: all in
  - equivalent: maximizing arithmetic mean
- Parlayed bets: Kelly criterion
  - equivalent: maximizing geometric mean
- What happens with all-in for parlayed bets?
- Note: payout asymmetry most important
- **Make sure your capital survives before it can compound**

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Section 2

The Kelly Criterion

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Some History

- Developed by J. L. Kelly at Bell Labs; Shannon reviewed
  - Texan tough guy, gunslinger, daredevil pilot and mathematician!
- Wirelines were used to transmit information between bookies
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Recommended read: William Poundstone, "Fortune's Formula", 2005, a layman version of the story behind the Kelly criterion, Shannon's forays into the casino and stock market, and Edward Thorp, a mathematician who figured out card counting for Blackjack and later ran a successful hedge fund Princeton Newport.

Formulation

- Assume m horses, each with i.i.d. probability of winning p_i
- Assume starting capital $S_0 = 1$
- Odds: o_i, alternative $(1 + r_i)$, r_i the rate of return
- Play for T races
  - allocate b_i fraction of capital on horse i
  - capital at T: $S_T = \prod_{t=1}^{T} \prod_{i=1}^{m} b_i o_i$
- Objective: assuming fully invested, choose allocation $b_i \geq 0$, $\sum_i b_i = 1$ to maximize S_T

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Maximising Wealth Growth

- Assume $T \to \infty$
  - maximise $E[\sum_{i=1}^{m} \log b_i o_i]$ subject to constraints
  - doubling rate: $W(\mathbf{b}, \mathbf{p}) := \sum_{i=1}^{m} p_i \log b_i o_i$
- Solution: the Kelly criterion, or log-optimal wealth growth
  - answer: $b_i^\star = p_i$, proportional gambling (for fair odds)
  - solve using standard KKT conditions, or log-sum inequality
- Nature of solution will depend on odds: see [CT91, Exercise 6.2]

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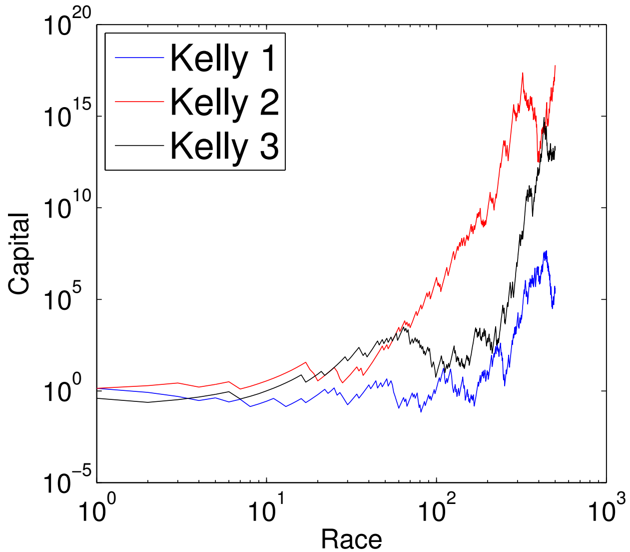
Odds can be classified according to $i\lambda := \sum_{i=1}^{m} \frac{1}{o_i}$. If $\lambda < 1$ these are *superfair odds*, $\lambda = 1$ are *fair odds* and $\lambda > 1$ are *subfair odds*. For subfair odds, proportional betting doesn't apply as some odds may be so poor that the criterion tells us not to bet. The solution is found via a water filling algorithm. The bottomline, however, is that the Kelly criterion only tells us to bet when the odds are favourable, otherwise don't bet.

Example Run of Kelly's Strategy



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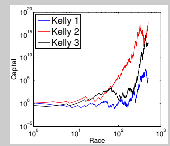


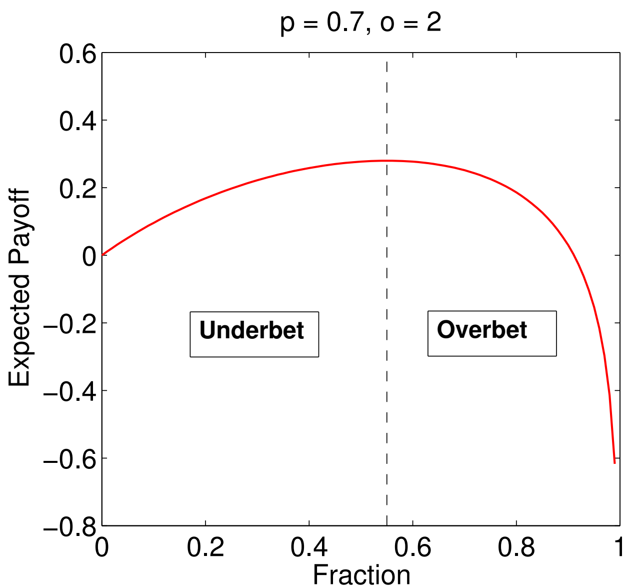
A Simple Bet

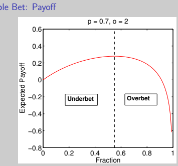
- Say a biased coin toss, win if heads, lose if tails
  - heads with probability p, q otherwise
  - each round, add \$1 to bet
- Odds: o-for-1 (remember: win-lose event)
- Kelly solution: $b^\star = \frac{op-q}{o} = \frac{p(o+1)-1}{o}$
  - what does it mean if $o = q/p$?
  - what does it mean when $b^\star < 0$ $(o < q/p)$?
  - what about $b^\star > 1$?
- A simple way to remember (for two events)

$$b^\star = \frac{\text{edge}}{\text{odds}}$$

The case $b^\star > 1$ can occur when the Kelly criterion is applied to odds coming from a continuous probability distribution.

Simple Bet: Payoff





Simple Bet: Under and Overbetting

- There is no gain in overbetting: growth decreases, risk increases
- Sweet spot: full Kelly for maximum wealth growth
- In practice, partial Kelly more applicable, i.e. αb_i^*
  - with α fraction, only α^2 volatility
  - more robust to error in estimating returns
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Section 3

Downsides

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Downsides

Caveats

- Strategy is guaranteed to beat any other strategy on wealth growth
- BUT Strategy is asymptotically optimal: assume playing forever
- No guarantee to win in the short term (or at all), just the best chance
- Psychologically unsettling: imagine capital dropping 60% right before tripling!
  - partial Kelly strategies trade smoothness with growth rate
- Guaranteed not to go to ruin
  - BUT assumes capital infinitely divisible
  - capital could be 10^{-10} but hey, at least not bankrupt!
  - can show $\lim_{T\to\infty} P(S_T > \epsilon) = 0$, for any $\epsilon > 0$
- Assumes know the probability of winning: not true in real life
  - again, half Kelly strategies help: gives a safety margin
  - estimation methods (e.g. maximum entropy, shrinkage)

A link between shrinkage estimation and why half Kelly strategies work well in practice is the paper by Rising and Wyner, "Partial Kelly Portfolios and Shrinkage Estimators", 2012 IEEE Symposium on Information Theory, pages 1618–1622.

Criticism from Modern Finance

- Kelly criterion assumes maximizing growth rate exponent
- Called the log-utility function in finance
- **Criticism 1**: not everybody would want to maximise growth rate exponent
  - does not take into account risk-averseness (or "sleep test")
  - definition of risk in finance: volatility
  - different utilities for different folks
- **Criticism 2**: time horizon, as discussed, need very long term
- Counter-argument: not many people want to do with less money
- "Money can't buy you happiness, but love can't get you a Ferrari."

Approximation of the Stock Market

- Suppose m risky assets, each with random "odds" r_i in one investment period
- One asset with return r_0 is deterministic
- Assume starting capital $S_0 = 1$
- The return vector $\mathbf{r}$, with $\boldsymbol{\mu}_{\mathbf{r}} = E[\mathbf{r}]$, $\Sigma = E[(\mathbf{r} - r_0\mathbf{1})(\mathbf{r} - r_0\mathbf{1})^T]$
  - Σ is full rank
  - correlations apply only "spatially"
- Derive the optimal allocation $\mathbf{b}$ to optimise the wealth doubling rate
  - optimise $E[\log(r_0 + \mathbf{b}^T(\mathbf{r} - r_0\mathbf{1}))]$
- Assume no constraints on $\mathbf{b}$
- For what return distribution is this allocation optimal?

Hint: use a Taylor series expansion of the objective around $\mathbf{b}_0 = (0, 0, \cdots, 0)$ to form a quadratic optimisation problem. r_0 is generally chosen to have a zero risk return, known as the *risk free rate*.

Further reading I

Thomas M. Cover and Joy A. Thomas, *Elements of information theory*, John Wiley and Sons, 1991.