Transform Methods & Signal Processing
Class Exercise 0

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The goal of this class exercise is to assess your grasp of the assumed knowledge for this course (Level II Fourier Series and Differential Equations or the equivalent). No marks will be assigned, but this is nevertheless an important exercise.

1. Trigonometry:

(a) simplify the following

i. $\cos^2(x) + \sin^2(x)$
Solution: $\cos^2(x) + \sin^2(x) = 1$

ii. $\sin(x + \pi/2)$
Solution: $\sin(x + \pi/2) = \cos(x)$

iii. $\sin(\alpha x + \pi)$
Solution: $\sin(\alpha x + \pi) = -\sin(\alpha x)$

iv. $\cos(x - 2\pi)$
Solution: $\cos(x - 2\pi) = \cos(x)$

(b) write the following without any products of cosines or sines

i. $2\cos(nx) \times \cos(mx)$
Solution: $2\cos(nx) \times \cos(mx) = \cos((n-m)x) + \cos((n+m)x)$

ii. $2\sin^2(\theta)$
Solution: $2\sin^2(\theta) = 1 - \cos(2\theta)$

2. Differentiation and Integration:

(a) Differentiate the following functions

	function	**derivative**
(i)	$\int e^{\alpha x}\, dx$	$f'(x) = \alpha e^{\alpha x}$
(ii)	$f(x) = x^2 \cos(-x)$	$f'(x) = 2x\cos(-x) + x^2 \sin(-x)$
(iii)	$f(x) = \sin(\ln(x))$	$f'(x) = \cos(\ln(x))/x$
(iv)	$f(x) = \sum_{i=0}^{\infty} x^i$, for $\lvert x\rvert < 1$	$f'(x) = \sum_{i=0}^{\infty} i x^{i-1} = 1/(1-x)^2$
(v)	$x^{1/\ln(x)}$	Note that $x^{1/\ln(x)} = \exp(\ln(x^{1/\ln(x)}))$ $= \exp(\ln(x)/\ln(x))$ $= \exp(1)$ Given that the function is a constant, then the derivative must be zero.

(b) Find the integrals

i. $\int e^{\alpha x}\, dx$

Solution: $\int e^{\alpha x}\, dx = e^{\alpha x}/\alpha + c$

ii. $\int_0^1 x e^{-ix}\, dx$

Solution: $\int_0^1 x e^{-ix}\, dx = i[xe^{-ix}]_0^1 - i\int_0^1 e^{-ix}\, dx = ie^{-i} + [e^{-ix}]_0^1 = ie^{-i} + e^{-i} - 1$

iii. $\int_{-\infty}^{\infty} e^{-\pi t^2} e^{-i2\pi st}\, dt$

Solution:

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\pi t^2} e^{-i2\pi st}\, dt &= \int_{-\infty}^{\infty} e^{-\pi\left(t^2 + i2st\right)}\, dt \\
&= e^{-\pi s^2} \int_{-\infty}^{\infty} e^{-\pi(t+is)^2}\, dt \\
&= e^{-\pi s^2} \int_{-\infty+is}^{\infty+is} e^{-\pi u^2}\, du \\
&= e^{-\pi s^2}
\end{aligned}$$

iv. $\int_{-\pi}^{\pi} \cos(nx)\, \cos(mx)\, dx$

Solution: Use the results of Q1.b.i, so for $n, m \in I\!N$, $n \neq m$

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(nx)\, \cos(mx)\, dx &= \frac{1}{2}\int_{-\pi}^{\pi} \cos((n-m)x) + \cos((n+m)x)\, dx \\
&= \frac{1}{2}\left[-\frac{\sin((n-m)x)}{(n-m)} - \frac{\sin((n+m)x)}{(n+m)}\right]_{-\pi}^{\pi} \\
&= 0
\end{aligned}$$

For $n, m \in I\!N$, $n = m$

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos^2(nx)\, dx &= \frac{1}{2}\int_{-\pi}^{\pi} 1 + \cos(2nx)\, dx \\
&= \frac{1}{2}\left[x - \frac{\sin(2nx)}{2n}\right]_{-\pi}^{\pi} \\
&= \pi
\end{aligned}$$

v. $\int_0^2 \int_0^4 (x^2 - y^2)\, dx\, dy$

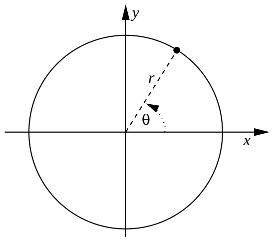
Solution:

$$\begin{aligned}
\int_0^2 \int_0^4 (x^2 - y^2)\, dx\, dy &= \int_0^2 \left[x^3/3 - y^2 x\right]_0^4 dy \\
&= \int_0^2 4^3/3 - 4y^2\, dy \\
&= \left[4^3 y/3 - 4y^3/3\right]_0^2 \\
&= 2 \times 4^3/3 - 4 \times 2^3/3 \\
&= 2^3(16-4)/3 \\
&= 2^7
\end{aligned}$$

3. **Co-ordinate systems:** Take circular co-ordinates r, θ such that

$$x = r\cos\theta, \;\; y = r\sin\theta$$

(a) draw a picture to illustrate this coordinate system
Solution:



(b) calculate the Jacobian J of the coordinate transform
Solution: The Jacobian Matrix is

$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

The Jacobian Determinant is

$$J = r\cos^2\theta + r\sin^2\theta = r.$$

Either might be abbreviated to Jacobian but the determinant will be more useful for us.

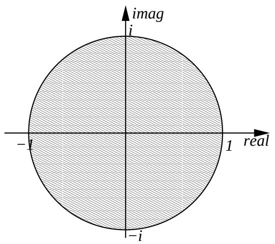
4. **Complex numbers:**

(a) simplify $(2+i)\times(1-3i)$
Solution: $(2+i)\times(1-3i) = 2.1 - 2.3i + i.1 - 3i^2 = 2 - 6i + i + 3 = 5 - 5i$.

(b) find the imaginary part $\Im\left(\frac{2+i}{5-3i}\right)$
Solution: $\frac{2+i}{5-3i} = \frac{2+i}{5-3i} \times \frac{5+3i}{5+3i} = \frac{(2+i)(5+3i)}{25+9} = \frac{10+5i+6i-3}{34} = \frac{7+11i}{34}$ The imaginary part of this is $11/36$.

(c) find the complex conjugate of $a + bi$
Solution: $a - bi$

(d) draw a picture of the set $|z| \leq 1$



Solution:

(e) find all of the values of $\sqrt[4]{-1}$
Solution: $\pm(1+i)/\sqrt{2}, \pm(1-i)/\sqrt{2}$

(f) for complex number z find $\frac{d}{dz}e^z$
Solution: Write $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$. Imagine a co-ordinate transform, consistent with our definition of $z = x + iy$, i.e.,

$$\begin{aligned} z &= x + iy \\ w &= x - iy \end{aligned}$$

Now the inverse transform would be

$$\begin{aligned} x &= \frac{z+w}{2} \\ y &= \frac{z-w}{2i} \end{aligned}$$

The chain rule is

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} \\ &= \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2i}\frac{\partial}{\partial y} \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{dz}e^z &= \frac{1}{2}\frac{\partial e^x}{\partial x}(\cos(y) + i\sin(y)) + \frac{1}{2i}\frac{\partial}{\partial y}(\cos(y) + i\sin(y)) \\ &= \frac{1}{2}e^x(\cos(y) + i\sin(y)) + \frac{e^x}{2i}(-\sin(y) + i\cos(y)) \\ &= e^x(\cos(y) + i\sin(y)) \\ &= e^z. \end{aligned}$$

(g) show $e^{z_1}e^{z_2} = e^{z_1+z_2}$ where $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$
Solution: Note first that $e^{z_1} = e^{x_1+iy_1}$ which we define to be $e^{z_1} = e^{x_1}e^{iy_1 i}$, and so

$$\begin{aligned} e^{z_1}e^{z_2} &= e^{x_1+y_1 i}e^{x_2+y_2 i} \\ &= e^{x_1}e^{x_2}e^{y_1 i}e^{y_2 i} \\ &= e^{x_1+x_2}e^{(y_1+y_2)i} \\ &= e^{x_1+x_2+(y_1+y_2)i} \\ &= e^{z_1+z_2} \end{aligned}$$

5. **Fourier series:**

(a) which of the following functions *odd*, *even* or *neither*

$$x^2, \quad e^x, \quad e^{|x|}, \quad x\sin(nx)$$

Solution:

$$even, neither, even, even$$

(b) given a periodic function (period 4), defined by $f(x) = x$ on the interval $[-2, 2]$

i. draw this function over the interval $[-6, 6]$
Solution: see the red "saw-toothed" curve in the following plot.

ii. obtain the Fourier series representation of $f(x)$.
Solution: For a function with period L

$$f(x) = \sum_{i=-\infty}^{\infty} A_n e^{i2\pi nx/L}$$

where for $n \neq 0$

$$\begin{aligned} A_n &= \frac{1}{L}\int_{-L/2}^{L/2} f(x) e^{-i2\pi nx/L}\,dx \\ &= \frac{1}{L}\int_{-L/2}^{L/2} x e^{-i2\pi nx/L}\,dx \\ &= \frac{1}{L}\left[x e^{-i2\pi nx/L}/(-i2\pi n/L)\right]_{-L/2}^{L/2} + \frac{1}{i2\pi n}\int_{-L/2}^{L/2} e^{-i2\pi nx/L}\,dx \\ &= \frac{L}{-2i\pi n}\left[\frac{e^{-i\pi n}+e^{i\pi n}}{2}\right] + \frac{1}{i2\pi n}\left[e^{-i2\pi nx/L}/(-i2\pi n/L)\right]_{-L/2}^{L/2} \\ &= \frac{L\cos(\pi n)}{-2i\pi n} + \frac{L}{2i(\pi n)^2}\left[\frac{e^{-i\pi n}-e^{i\pi n}}{-2i}\right] \\ &= \frac{(-1)^n L}{-2i\pi n} + \frac{L\sin(\pi n)}{2i(\pi n)^2} \\ &= \frac{(-1)^n L}{-2i\pi n} \end{aligned}$$

A_0 is a special case and $A_0 = 0$. If we take the two components

$$A_n e^{i2\pi nx/L} + A_{-n} e^{-i2\pi nx/L} = \frac{L(-1)^{(n+1)}}{\pi n}\frac{e^{i2\pi nx/L} - e^{-i2\pi nx/L}}{2i} = \frac{L(-1)^{(n+1)}\sin(2\pi nx/L)}{\pi n}$$

we get

$$f(x) = \sum_{n=1}^{\infty} \frac{L(-1)^{(n+1)}\sin(2\pi nx/L)}{\pi n},$$

which is the standard Fourier series representation of the saw-tooth wave form $f(x)$.
The following figure illustrates the Fourier series construction of the saw tooth, with each graph showing the sum from $n = 1, \ldots, N$ for increasing $N = 1, \ldots, 7$. Notice how as more components of the Fourier series are included, the blue curve gets closer to the red curve (the original function $f(x)$).