Transform Methods & Signal Processing
Class Exercise 3: solutions

Matthew Roughan
<matthew.roughan@adelaide.edu.au>

1. [7 marks] Find (by hand) the DFT of the following signals (using the simplest possible method you can think of), and show that the Rayleigh-Parceval Theorem holds for these signals.

Solution: Note the DFT is defined by

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-i2\pi kn/N}$$

(a) Calculate $\mathcal{F}\{(0,0,1,1)\}$ directly by taking

$$\begin{array}{lllllll} X(0) & = & e^{-i2\pi 0/4} + e^{-i2\pi 0/4} & = & 1+1 & = & 2 \\ X(1) & = & e^{-i2\pi 2/4} + e^{-i2\pi 3/4} & = & e^{-i\pi} + e^{-i\pi 3/2} & = & -1+i \\ X(2) & = & e^{-i2\pi 4/4} + e^{-i2\pi 6/4} & = & e^{-i2\pi} + e^{-i3\pi} & = & 1-1=0 \\ X(3) & = & e^{-i2\pi 6/4} + e^{-i2\pi 9/4} & = & e^{-i3\pi} + e^{-i\pi 9/2} & = & -1-i \end{array}$$

(b) There are several ways to calculate $\mathcal{F}\{(1,0,-1,0,1,0,-1,0,1,0,-1,0,1,0,-1,0)\}$

- Note that the terms are $\cos(2\pi(1/4)n)$, and use the fact that the FT of cos is the discretized version of that given in question 1.
- Note that this is the signal $(1,0,-1,0)$ repeated 4 times, and that $\mathcal{F}\{(1,0,-1,0)\} = (0,2,0,2)$ by direct computation as in part (a). Duality implies that if we upsample the FT (by inserting zeros), we repeat the time series. Hence, when we repeat the time series 4 times, we need to insert 3 zeros between each value in the FT. The scale of the values will also be multiplied by 4.
- Note that this is the signal $(1,-1,1,-1,1,-1,1,-1)$, upsampled by interleaving with zeros. Hence, we should see its spectrum repeat twice in the Fourier Domain. We can directly compute $\mathcal{F}\{(1,-1,1,-1,1,-1,1,-1)\} = (0,0,0,0,8,0,0,0)$ or use an argument similar to the previous approach.

The result is $\mathcal{F}\{(1,0,-1,0,1,0,-1,0,1,0,-1,0,1,0,-1,0)\} = (0,0,0,0,8,0,0,0,0,0,0,0,8,0,0,0)$

(c) We could compute $\mathcal{F}\{(1,2,1,0)\}$ as before, but it is much simpler to note that $(1,2,1,0) = (0,0,1,1)*(0,0,1,1)$ where $*$ denotes a circular convolution. We already know that $\mathcal{F}\{(0,0,1,1)\} = (2,-1+i,0,-1-i)$. Given $y = x*x$ the FTs will be

$$Y(k) = X(k)^2$$

So

$$\mathcal{F}\{(1,2,1,0)\} = (4,-2i,0,2i)$$

(d) As before, we have a signal that replicates $(1,-i,1,-i)$ exactly 2 times. Using a similar derivation, the result is

$$\mathcal{F}\{(1,i,-1,-i,1,i,-1,-i)\} = (0,0,8,0,0,0,0,0)$$

2. [3 marks] A new type of high-quality digital audio media is proposed. To convert to this media, an analogue audio signal is passed through a low-pass filter, and then sampled at 50 kHz, with a 24 bit fixed point representation of the samples.

(a) What should be the stop-band of the low-pass filter?

Solution: To prevent aliasing the low-pass needs to remove frequencies above $f_s/2$, which means its stop-band should be 25 kHz.

(b) If the filter is intended to reduce aliasing noise to the same level as the quantization noise (assuming a full load, with no clipping), then what should be its stop-band attenuation?

Solution: We reduce quantization noise by approximately 6dB per bit for a fixed point representation. Of the 24 bits, 1 is used for the sign, and 23 for the value, so the quantization noise (for a full load with no clipping) should be at approximately $-23 \times 6 = -138dB$. To reduce aliasing noise by this much, the stop-band attenuation would have to be $-138dB$.

(c) Given a sample 0.1 seconds long, what would be the lowest resolvable frequency?

Solution: The number of samples in 0.1 seconds is $N = 5000$, and the sampling frequency is $f_s = 50,000$. The resolution in the transform domain is $f_s/N = 10Hz$, so the lowest resolvable frequency would be 10 Hz.

3. [5 marks] Obviously there is no correct answer to this question. However, there are some things I am looking for in an answer. Firstly, a clear, concise answer.Secondly, you need to make the connection between dynamic range, as portrayed in the articles, and in our lectures. The critical point is that CD audio uses 16 bits, with a resulting 96dB dynamic range, which is significantly more than was available in consumer analogue technologies. Dynamic range of a technology specifies the ratio of the loudest to the softess possible sounds that can be reconstructed. However, due to "fashion" this available dynamic range is being used less and less. The ratios of the loudest to the softest sounds actually recorded is much smaller than the possible range. This is a really strange phenomena, because dynamic range should make music sound better, but there is a recently developed belief that this is wrong.

In terms of calculating the RMS signal power, there are a number of things to remember. Firstly, in Matlab, when we use `wavread` the resulting signal will typically be in $[-1,1]$ (though care must be taken as this may change for different input files.) So the amplitude of the largest possible unclipped sine wave will be 1, and its RMS power will be $1/\sqrt{2}$, so we need to compare measured RMS powers to $1/\sqrt{2}$. The results should be measured in dB so the calculation we should perform might look like

$$RMS = 20\log_{10}\sqrt{\frac{2}{N}\sum_{i=1}^{N} x_i^2}.$$

though there are acceptable variants on this.

A minor issue is that most music today is recorded in stereo. We can report simply the average of the two results, or preferably report both numbers.

Figure **??** shows a summary of the results found from the class in 2009.

4*. [5 marks] Prove that the Fourier transform of a Dirac comb (or Dirac train) with unit spacing is a Dirac comb with unit spacing.

Solution: A simple way to see this is to note that the Dirac comb is a periodic function (though it isn't strictly a function), with period one, and so we could compute a Fourier series for it by

$$d(t) = \sum_{n=-\infty}^{\infty} A_n e^{i2\pi nt}$$

where A_n is given by ($L = 1$ is the period)

$$\begin{array}{lll} A_n & = & \frac{1}{L}\int_{-L/2}^{L/2} \delta(t) e^{-i2\pi nt/L}\, dt \\ & = & 1 \end{array}$$

so

$$d(t) = \sum_{n=-\infty}^{\infty} e^{i2\pi nt}$$

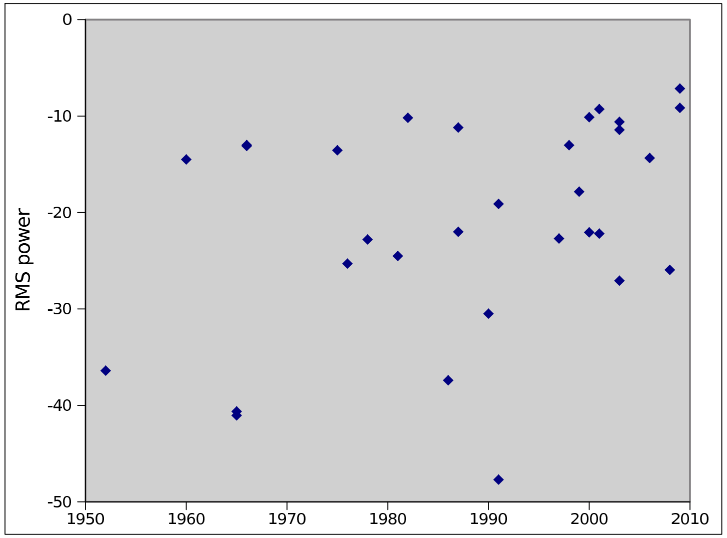

Figure 1: Summary of the results found from the class in 2009 for RMS power of audio tracks. Although the results are highly variable (possibly because of errors in calculation) a general trend is observable.

Now take the Fourier transform of $d(t)$ and we get

$$\begin{aligned} \mathcal{F}\{d(t)\} &= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(t-k)\, e^{-i2\pi st}\, dt \\ D(s) &= \sum_{k=-\infty}^{\infty} e^{-i2\pi ks} \end{aligned}$$

which we saw above was just a Dirac comb with unit spacing, so

$$D(s) = \mathcal{F}\{d(t)\} = d(s),$$

where $d(s)$ is again just the Dirac comb with unit spacing.