Examination in School of Mathematical Sciences

Semester 2, 2006

9694	TRANSFORM METHODS AND SIGNAL PROCESSING
	APP MATH 4043

Official Reading Time:	10 mins
Writing Time:	180 mins
Total Duration:	190 mins

NUMBER OF QUESTIONS: 5 TOTAL MARKS: 100

Instructions

- Answer ALL questions.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

Materials

- 1 Blue books are provided.
- Calculators ARE permitted.
- Open book examination.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO

1. (a) State the definition of the continuous Fourier transform, and

    1. derive (from the above definition) the FT of a rectangular pulse $r(t)$ defined by $r(t) = u(t + 1/2) - u(t - 1/2)$ where $u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$.
    2. derive (using properties of the continuous FT) the FT for $r(t)e^{-\pi t^2/\sigma^2}$ where $r(t)$ is the rectangular pulse.
    3. now consider the limit as $\sigma \to \infty$, what would the FT from part 2 converge to?

   (b) State the definition of the discrete Fourier transform, and then

    1. derive the DFT of $(1, 0, 0, 0)$
    2. from the DFT of $(1, 0, 0, 0)$ derive the DFT of $(1, 0)$ (and state the property of the DFT that you use).
    3. from the DFT of $(1, 0)$ derive the DFT of $(0, 1)$ (and state the property of the DFT that you use).
    4. from the DFT of $(0, 1)$ derive the DFT of $(0, 1, 0, 1)$ (and state the property of the DFT that you use).
    5. from the two DFTs of $(1, 0, 0, 0)$ and $(0, 1, 0, 1)$, find the DFT of $(3, 6, 0, 6)$.

   (c) Give the Haar wavelet coefficients on the dyadic grid for a signal

    1. $(1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0)$

   at octaves $j = 1, 2$ and 3.

[20 marks]

Solutions:

(a) [**6 marks**] $F(s) = \int_{-\infty}^{\infty} f(t)e^{-i2\pi st}\,dt$

1. Starting from the definition of the FT we get

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} r(t)e^{-i2\pi st}\,dt \\ &= \int_{-1/2}^{1/2} e^{-i2\pi st}\,dt \\ &= \int_{-1/2}^{1/2} \cos(2\pi st) + i\sin(2\pi st)\,dt \end{aligned}$$

The integral of the odd function (sin) will be zero, and so we only need consider the integral of the $\cos$ function, i.e.

$$\begin{aligned} F(s) &= \int_{-1/2}^{1/2} \cos(2\pi st)\, dt \\ &= \left[\frac{\sin(2\pi st)}{2\pi s}\right]_{-1/2}^{1/2} \\ &= \frac{2\sin(\pi s)}{2\pi s} \\ &= \frac{\sin(\pi s)}{\pi s} \\ &= \text{sinc}(s) \end{aligned}$$

2. The FT of a product (in the time domain) is a convolution in the frequency domain. The FT of a Gaussian $e^{-\pi t^2}$ is a Gaussian $e^{-\pi s^2}$. Scaling by $1/\sigma$ results (through the scaling property) in a FT of

$$\mathcal{F}\left\{e^{-\pi t^2/\sigma^2}\right\} = e^{-\pi s^2 \sigma^2}/\sigma.$$

The FT of a rectangular pulse is a sinc function, so the complete FT is

$$\mathcal{F}\left\{r(t)e^{-\pi t^2/\sigma^2}\right\} = \text{sinc}(s) * e^{-\pi s^2\sigma^2}/\sigma.$$

3. In the limit as $\sigma \to \infty$ the Gaussian in the time-domain converges to a constant, and so the Gaussian ($e^{-\pi s^2\sigma^2}/\sigma$) in the frequency domain converges to a delta function. The convolution of a delta function with $f(t)$ is just $f(t)$, and so the FT becomes a simple sinc function.

(b) [**10 marks**]

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-i2\pi kn/N}$$

1. We can calculate the DFT of $(1, 0, 0, 0)$ directly from the definition as follows

$$\begin{aligned} X(0) &= e^{-i2\pi 0/4} &= 1 \\ X(1) &= e^{-i2\pi 0/4} &= 1 \\ X(2) &= e^{-i2\pi 0/4} &= 1 \\ X(3) &= e^{-i2\pi 0/4} &= 1 \end{aligned}$$

and the result is

$$\mathcal{F}\{(1, 0, 0, 0)\} = (1, 1, 1, 1).$$

2. To obtain the FT of $(1, 0)$, we note that $(1, 0, 0, 0)$ is a padded version of $(1, 0)$, and so its Fourier transform of can be obtained by sampling that of $(1, 0, 0, 0)$, i.e.

$$\mathcal{F}\{(1, 0)\} = (1, 1).$$

3. To obtain the FT of $(0,1)$, note that this is a time shifted version of $(1,0)$, and so we can obtain its FT by a simple phase change in the frequency domain, i.e. by multiplying the FT by a complex exponential $e^{-i2\pi k/2}$ and the resulting FT is

$$\mathcal{F}\{(0,1)\} = (1,-1).$$

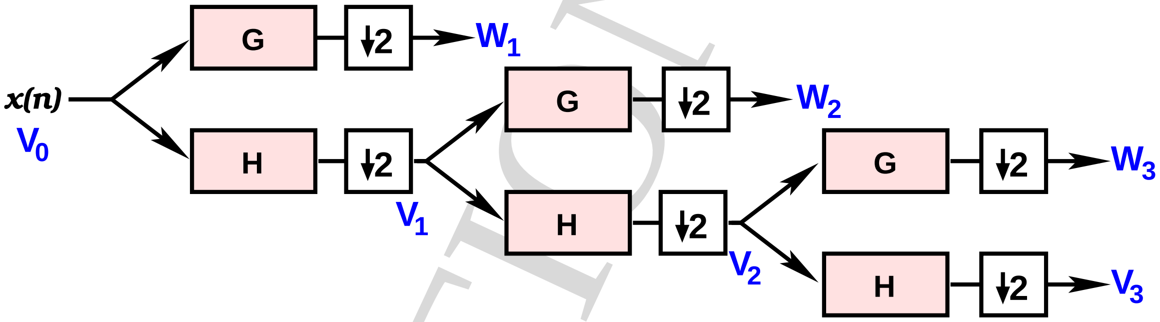
4. The signal $(0,1,0,1)$ is simply the signal $(0,1)$ repeated twice, and so we could obtain it by inserting zeros into the spectrum of $(0,1)$ (and scaling), i.e.

$$\mathcal{F}\{(0,1,0,1)\} = (2,0,-2,0).$$

5. The FT is a linear operator, and so the FT of $(3,6,0,6) = 3(1,0,0,0) + 6(0,1,0,1)$ is given by

$$cF(3,6,0,6) = cF3(1,0,0,0) + 6(0,1,0,1) = 3(1,1,1,1)+6(2,0,-2,0) = (15,3,-9,3).$$

(c) [**4 marks**] The discrete filters for the Haar wavelet transform are $h = (1,1)/\sqrt{2}$, and $g = (1,-1)/\sqrt{2}$, but for the purpose of simplifying the computations, we shall not use the scaling factor of $\sqrt{2}$ until the end. The pyramidal wavelet algorithm looks like



Each block refers to the use of one of the filters defined above followed by a downsampling, and each filtering corresponds to a convolution, for instance, the two first filters result in

$$\begin{aligned} x * g \downarrow 2 &= \sum_{m=-\infty}^{\infty} h(m-2n)x(m) \\ x * h \downarrow 2 &= \sum_{m=-\infty}^{\infty} g(m-2n)x(m) \end{aligned}$$

The first term is the wavelet details $d_1(n)$ (at octave $j = 1$), and the second term is the approximation $a_1(n)$ (at octave $j = 1$). So we write

$$\begin{aligned} a_1(n) &= \sum_{m=-\infty}^{\infty} h(m-2n)x(m) \\ d_1(n) &= \sum_{m=-\infty}^{\infty} g(m-2n)x(m) \end{aligned}$$

Now (ignoring the scale factors) $h(0) = 1$ and $h(1) = 1$, so

$$\begin{aligned}
a_1(0) &= h(0)x(0) + h(1)x(1) \\
&= x(0) + x(1) \\
&= 2 \\
a_1(1) &= h(0)x(2) + h(1)x(3) = 2 \\
a_1(2) &= h(0)x(4) + h(1)x(5) = 0 \\
a_1(3) &= h(0)x(6) + h(1)x(7) = 0 \\
a_1(4) &= h(0)x(8) + h(1)x(9) = 0 \\
a_1(5) &= h(0)x(10) + h(1)x(11) = 2 \\
a_1(6) &= h(0)x(12) + h(1)x(13) = 0 \\
a_1(7) &= h(0)x(14) + h(1)x(15) = 0
\end{aligned}$$

So $a_1 = (2, 2, 0, 0, 0, 2, 0, 0)$. Notice there are half as many terms as the original signal because we have downsampled by 2. Similarly $g(0) = 1$ and $g(1) = -1$ so

$$\begin{aligned}
d_1(0) &= g(0)x(0) + g(1)x(1) \\
&= x(0) - x(1) \\
&= 0 \\
d_1(1) &= g(0)x(2) + g(1)x(3) = 0 \\
d_1(2) &= g(0)x(4) + g(1)x(5) = 0 \\
d_1(3) &= g(0)x(6) + g(1)x(7) = 0 \\
d_1(4) &= g(0)x(8) + g(1)x(9) = 0 \\
d_1(5) &= g(0)x(10) + g(1)x(11) = 0 \\
d_1(6) &= g(0)x(12) + g(1)x(13) = 0 \\
d_1(7) &= g(0)x(14) + g(1)x(15) = 0
\end{aligned}$$

So $d_1 = (0, 0, 0, 0, 0, 0, 0, 0)$, so the approximation at octave 1 is perfect (which we could see from the signal).

Given the pyramidal structure, we can then write the following relationships between each of the larger octaves.

$$\begin{aligned}
a_{j+1}(n) &= \sum_{m=-\infty}^{\infty} h(m-2n)a_j(m) = \left[a_j * \bar{h}\right](2n) \\
d_{j+1}(n) &= \sum_{m=-\infty}^{\infty} g(m-2n)a_j(m) = \left[a_j * \bar{g}\right](2n)
\end{aligned}$$

If we use this to compute the second scale approximation we get

$$
\begin{aligned}
a_2(0) &= h(0)a_1(0) + h(1)a_1(1) \\
&= a_1(0) + a_1(1) \\
&= 4 \\
a_2(1) &= h(0)a_1(2) + h(1)a_1(3) = 0 \\
a_2(2) &= h(0)a_1(4) + h(1)a_1(5) = 2 \\
a_2(3) &= h(0)a_1(6) + h(1)a_1(7) = 0
\end{aligned}
$$

So $a_2 = (4, 0, 2, 0)$. Likewise to compute the d_2 we get

$$
\begin{aligned}
d_2(0) &= g(0)a_1(0) + g(1)a_1(1) \\
&= a_1(0) - a_1(1) \\
&= 0 \\
d_2(1) &= g(0)a_1(2) + g(1)a_1(3) = 0 \\
d_2(2) &= g(0)a_1(4) + g(1)a_1(5) = -2 \\
d_2(3) &= g(0)a_1(6) + g(1)a_1(7) = 0
\end{aligned}
$$

So $d_2 = (0, 0, -2, 0)$. We repeat to get the next higher scale.

$$
\begin{aligned}
a_3(0) &= h(0)a_2(0) + h(1)a_2(1) \\
&= a_2(0) + a_2(1) \\
&= 4 \\
a_3(1) &= h(0)a_2(2) + h(1)a_2(3) = 2 \\
d_3(0) &= g(0)a_2(0) + g(1)a_2(1) \\
&= a_2(0) - a_2(1) \\
&= 4 \\
d_3(1) &= g(0)a_2(2) + g(1)a_2(3) = 2
\end{aligned}
$$

The MRA up to octave 3 (based on the Haar wavelets) including the $\sqrt{2}$ factors is therefore given by $\{a_3, d_1, d_2, d_3\}$ where these are

$$
\begin{aligned}
a_3 &= (4, 2)/2^{3/2} \\
&= (\sqrt{2}, 1/\sqrt{2}) \\
d_3 &= (4, 2)/2^{3/2} \\
&= (2/\sqrt{2}, 1/\sqrt{2}) \\
d_2 &= (0, 0, -2, 0)/2 \\
&= (0, 0, -1, 0) \\
d_1 &= (0, 0, 0, 0, 0, 0, 0, 0)/\sqrt{2} \\
&= (0, 0, 0, 0, 0, 0, 0, 0)
\end{aligned}
$$

2. (a) The major audio components of speech lies below the range of 4kHz.
   1. What sampling rate would be required for such a signal?
   2. Describe the processing of the audio signal when converting it from analogue to digital.
   3. What would be the bit rate of the signal assuming we need at around 40dB dynamic range.

   (b) High capacity submarine cables are rather expensive, and so only get build in certain places. For other international links the typical approach is to use a satellite connection. Given limited satellite capacity (say between Australia and New Guinea), it would be desirable to compress the above signal such that its bit rate is substantially lower. Briefly describe one approach to compression that we have discussed in lectures (your answer should take no more than half a page of text, though you may supplement this with diagrams). Make sure that you explain the intuition behind the approach.

   (c) Give a reason that we might wish to upsample the above signal before Digital-to-Analogue conversion?

[20 marks]

Solutions:

(a) [**9 marks**]
1. We must sample at least as fast as the Nyquist rate in order to avoid aliasing. The Nyquist rateis twice the highest frequency present in the signal, i.e. 8 kHz.
2. We would first pass the signal through a low-pass filter to ensure that there are no components above the range of interest that could cause aliasing. We then sample the signal at the rate of 8 kHz, but also we quantize the signal so that its values are constrained to some finite set of quantization levels.
3. With uniform quantization the dynamic range of a signal is approximately 6dB per bit, and so to achieve 40 dB we need at least 7 bits. Given one bit is used for sign, we need 8 bits per sample. 8 bits per sample, with 8,000 samples per second results in 64kbps (this is the standard Pulse Code Modulation (PCM) rate for the uncompressed voice codec for a DS0 channel).

(b) [**8 marks**] Compression involves reducing the memory requirements for a set of data (e.g. our voice signal). There are two types of compression: lossless, and lossy, but only lossy compression is considered in this course. The aim off lossy compression is to reduce the number of bits required to store (or transmit) the signal, but with minimal *perceivable* change to the signal. For example, with our voice signal we would like to encode the signal with fewer bits, but with few audible artifacts. The key idea in many such compression schemes is that errors introduced in the frequency domain are often less easily perceived than errors in the time-domain. Hence, a typical approach to compression is to transform the signal (e.g. using a cosine transform), and quantise the signal (more coarsely) in the frequency domain. We are less sensitive to errors in certain parts of the spectrum, and so these parts can be quantized with fewer bits.

[extra credit: In audio signals, a phenomena called masking occurs. A loud signal in one frequency band can hide a small signal in another band, and so we can quantize adaptively in the frequency domain to take advantage of anyu masking that occurs].

(c) [**3 marks**] A typical reason for upsampling before performing Digital-to-Analogue conversion is that fitlering in the analogue domain is relatively expensive (and doesn't follow Moore's law, and so is unlikely to reduce significantly in cost). If we upsample in the digital domain, we can produce a signal which is a better approximation of the original signal before conversion, and so the analogue filters are not as challenged, and can hence be made more cheaply (for the same quality output).

3. (a) Show that the inverse Discrete Fourier transform applied to the Discrete Fourier Transform of a signal always gives us back the original signal (i.e. that it really is the inverse transform).

(b) What is the key insight of the FFT? Describe the advantage of the FFT over direct calculation of the DFT.

(c) What is the phenomena of leakage? How can we reduce leakage, and what tradeoffs do we encounter when trying to do so.

[20 marks]

Solutions:

(a) [**10 marks**] DFT is given by

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-i2\pi kn/N}$$

and we aim to show that the IDFT is given by

$$x(m) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{-i2\pi km/N}$$

Applying them both to a signal we get

$$\begin{aligned} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} x(n) e^{-i2\pi kn/N} e^{i2\pi km/N} &= \frac{1}{N} \sum_{n=0}^{N-1} x(n) \sum_{k=0}^{N-1} e^{-i2\pi kn/N} e^{i2\pi km/N} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} x(n) N \delta_{km} \\ &= x(m) \end{aligned}$$

The second step follows because when $m = n$

$$\begin{aligned} \sum_{k=0}^{N-1} e^{-i2\pi kn/N} e^{i2\pi km/N} &= \sum_{k=0}^{N-1} e^{-i2\pi kn/N} e^{i2\pi kn/N} \\ &= \sum_{k=0}^{N-1} 1 \\ &= N \end{aligned}$$

However, when $n \neq m$ the sum becomes

$$\begin{aligned} \sum_{k=0}^{N-1} e^{-i2\pi kn/N} e^{i2\pi km/N} &= \sum_{k=0}^{N-1} e^{i2\pi k(m-n)/N} \\ &= \sum_{k=0}^{N-1} e^{i2\pi kh/N} \end{aligned}$$

where h is some non-zero integer. Now, note that the points $e^{-i2\pi kh/N}$ are a set of uniformly spaced samples around the unit circle in the complex plane. Intuitively, we see that the complex and real components will cancel, and so the sum $= 0$.

(b) [**4 marks**] The key insight is "divide and conquer", i.e. by dividing the signal up into components (and using symmetries of the transform) we can speed up the transform, in particular when this approach is applied recursively. The advantage is that the algorithm's computational complexity is $(N \log N)$ as compared with $O(N^2)$ for the direct approach.

(c) [**6 marks**] Leakage occurs when we take the DFT of a finite set of data from a signal with non-integral period. There are a number of ways to explain it (one will be sufficient for the answer)

- Fourier coefficients can be viewed as narrow bandpass filters, which therefore have side-lobes that respond to the non-integral frequencies. Integral frequencies occur at the nulls of the side-lobe pattern.
- Taking the DFT of a finite signal implicitly assumes that the data is periodic, and so we could consider implicitly that we take the Fourier transform of repeats of the signal placed one after the other. Given a non-integral period signal, there will be transients introduces at the edge.
- We can view a finite data set as an infinite signal multiplied by a rectangular pulse (that truncates the data). The FT of the product will therefore be the convolution of the FT of the infinite signal and a sinc function.

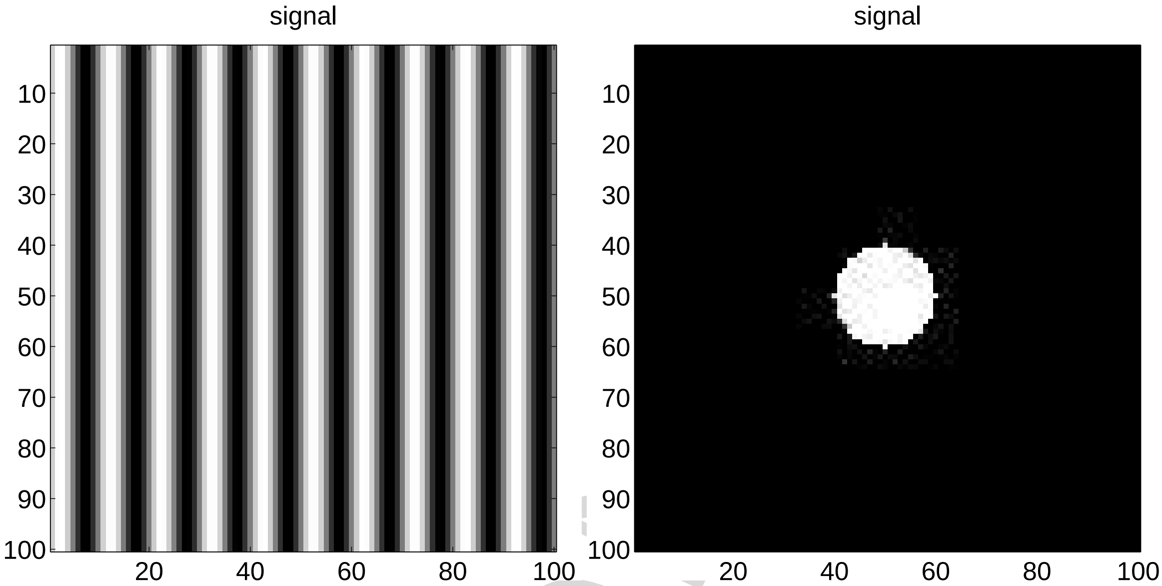
We can reduce leakage by using a better window function (than the rectangular window) to truncate the signal. Better functions will tend to have smoother edges to eliminate transients at the edges of the signal.

There are two obvious tradeoffs we have discussed

- Between width of the main lobe (and consequently resolution) and the size of the side-lobes (and consequently sensitivity).
- Between regularity of the window and decay of side-lobes in the Fourier domain.

4. 2D example

   (a) Look at the images displayed in figure below (the first is sinusoidal in one direction, and constant in the other, the second is zero outside, and one inside a circle). Describe what the power-spectrum of these images would look like.



   (b) We can define a 2D z-transform as follows:

$$G(z, w) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} g(i, j) z^{-i} w^{-j}$$

   Given a 2D z-transform of the following form

$$G(z, w) = (z^{-1} - z)(w^{-1} + 2 + w)$$

   describe the type of filter that would result.

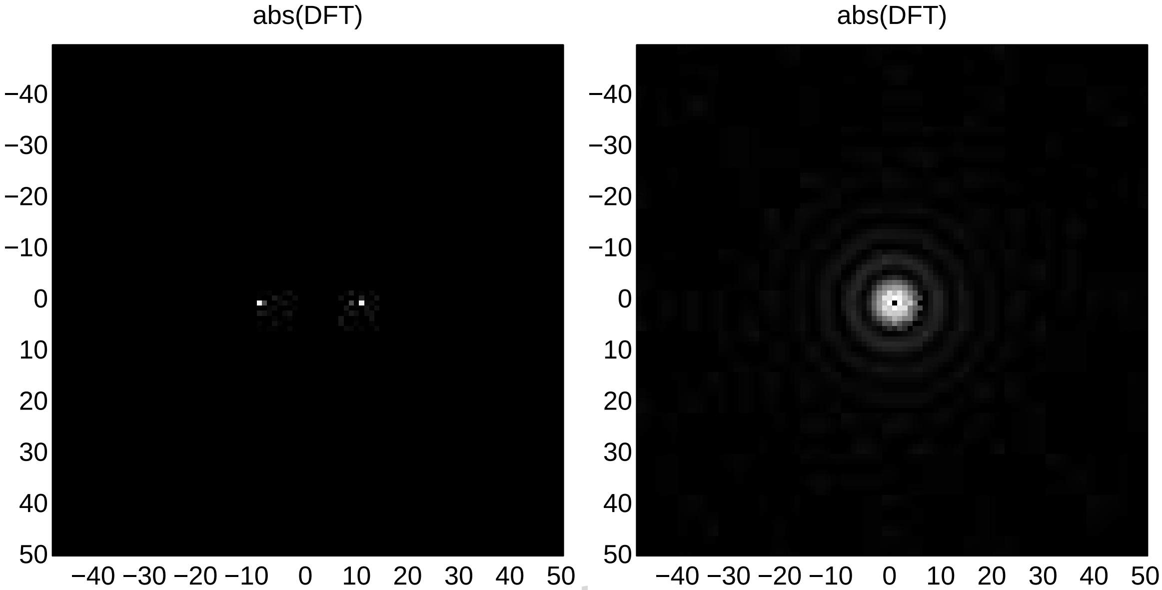
   (c) Describe "jaggies" in images, and how they relate to aliasing artifacts.

[20 marks]

Solutions:

(a) [**10 marks**] In the first case, the image shows a sinusoid in the x direction, and a constant in the y direction. Note that there are 10 repetitions of the sinusoid, so it is frequency 10. Hence, the power-spectrum will have a delta at the frequency bin corresponding to frequency 10 horizontally, and zero vertically, and the corresponding term for frequency -10. Tehe figure below shows this FT.

In the second case, the function is (approximately) radially symmetric, and so the FT will also have (approximate) radial symmetry. Further, if we took a single slice through the image (say at $y = 50$) we would see a profile that looked like a rectangular pulse. Therefore, we should expect to see the FT of a rectangular pulse (a sinc) when we examine a slice of the image's FT. Therefore the power-spectrum will look like a sinc^2 function rotated around the zero frequency point.



(b) [**7 marks**] We determine the filter directly by multiplying out all of the coefficients, and equating coefficients between the filter and the time-domain coefficients, i.e.,

$$\begin{aligned} G(z,w) &= (z^{-1}-z)(w^{-1}+2+w) \\ &= z^{-1}w^{-1}+2z^{-1}+wz^{-1}-zw^{-1}-2z-wz \end{aligned}$$

so the filter looks like

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{pmatrix}$$

Bonus marks for being a bit more clever and realising that the filter is a product (in the transform domain) and therefore corresponds to a convolution in the time domain). The two individual filters in the time domain should be obvious, e.g.

$$A_z = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad A_w = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Convolving these together we get A.

We can also note that because the filter separates out into filters in the x and y direction, we could implement as a cascade, and can therefore worry about the filter characteristics in each direction separately.

- the filter $(z^{-1}-z) = (1-z^2)/z$ has a pole at zero, and 2 zeros at 1 (in the complex plane. The zeros push down, and 1 corresponds to low-frequencies, so this is a high-pass filter.

- the filter $(w^{-1} + 2 + w) = (1 + w)^2/w$ has a pole at zero, and two zeros at -1 in the complex plane. The point -1 corresponds to high-frequencies, so this is a low-pass filter.

Other features of the filter – it is non-causal, time (space) invariant, and linear.

(c) [**3 marks**] Jaggies result when we sample an "edge" in some image and we see a staircase like effect along the edge. For instance, when we draw a straight line at an angle in an image.

They are related to aliasing because the "edge" of interest is *not* a bandlimited signal, and so when we sample at any finite rate, the signal will be aliased.

The visual impact of jaggies can be reduced by prefiltering the object to be sampled (typically using a low-pass filter) such that the signal being sampled is bandlimited, and so aliasing is not important.

5. Write a brief (less than one page) essay contrasting the Short-Time Fourier Transform and the Wavelet Transform. In particular, discuss how each deals with uncertainty bounds (and why this is important), and how the construction of the wavelet basis functions is helpful in this regard. Highlight why the Wavelet transform would often be a preferable.

   Please note that approximately 50% of marks will be based on content, and 50% on presentation, including clarity of your arguments. [20 marks]

   Solutions: Obviously no correct solution exists to an essay question. In marking the question I will take into account the

   - clarity of writing, and quality of presentation,
   - the following points should be covered
     - the uncertainty principle (definition and consequences)
     - the different ways the wavelet and break up the time frequency space
     - the construction of the wavlets from a mother wavelet by dilations and translations, and the discretization to the dyadic grid which result in a set of basis functions which break up the time-frequency domain in the required manner.
     - the fact that the wavelet's break up of the time-frequency domain is in many cases more desirable because it breaks the time domain into block of an appropriate resolution for the frequency under consideration.

   Note the page limit, and stick to it.