Variational Methods and Optimal Control
Class Exercise 4 solutions

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1. Solution: 6 marks

- The constraint is a **non-holonomic**.
- The Lagrange multiplier is $\lambda(t)$, and the new functional is

$$J\{y,z\} = \int_{x_0}^{x_1} y^2 + z^2 + \lambda(y' - z + y)\,dx$$

- The Euler-Lagrange equations are

$$\begin{aligned} \frac{\partial h}{\partial y} - \frac{d}{dx}\frac{\partial h}{\partial \dot{y}} &= 0 \\ \frac{\partial h}{\partial z} - \frac{d}{dx}\frac{\partial h}{\partial \dot{z}} &= 0 \\ \frac{\partial h}{\partial \lambda} - \frac{d}{dx}\frac{\partial h}{\partial \dot{\lambda}} &= 0 \end{aligned}$$

which give

$$\begin{aligned} 2y + \lambda - \lambda' &= 0 \\ 2z - \lambda &= 0 \\ y' - z + y &= 0 \end{aligned}$$

- The second equations gives

$$\lambda = 2z$$

substitute into the first equation and we get

$$2y + 2z - 2z' = 0$$

Differentiate and rearrange and we get

$$y' = -z' + z''$$

and we substitute these two into into the last equation to get

$$y' - z + y = -z' + z'' - z - z + z' = 0$$

which simplifies to the linear homogenous ODE

$$z'' - 2z = 0$$

This has solutions

$$z = c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x}$$

We also know

$$y = -z + z' = c_1(-1+\sqrt{2})e^{\sqrt{2}x} + c_2(-1-\sqrt{2})e^{-\sqrt{2}x}$$

which we can see satisfy the constraints.
We would need boundary conditions to determine the values of c_1 and c_2.

2. Given that

$$F\{y(x)\} = \int_0^1 (y'^2 - y^2)\,dx,$$

with the constraint on $y(x)$ that

$$\int_0^1 \sqrt{1+y'^2}\,dx = \sqrt{2},$$

and the end conditions $y(0) = 0$ and $y(1) = 1$, prove that $F\{y(x)\}$ achieves its minimum value for $y = x$.

Solution: 1 mark The thing to note in this problem is that the distance of a straight line from $(0,0)$ to $(1,1)$ is $\sqrt{2}$, and hence the only possible curve that meets the constraint is a straight line.

As there is no possible variation, we cannot use the Euler-Lagrange equations, and the solution is the **rigid** extremal, $y = x$.

3. The maximum entropy principle is an extension of Laplace's principle of insufficient reason, which in essence says we should not assume things that are not supported by evidence. For instance, in probability, unless we have reason to suspect otherwise, we would assume events are equally likely, e.g., the probability of heads coming up on a coin toss is 1/2.

Maximum entropy extends this by noting that if we maximize the (Shannon) entropy of a probability distribution constrained by the facts we know about the distribution, we will derive the estimate of that distribution which makes the least assumptions about the distribution that aren't supported by the data.

The Shannon entropy of a distribution with two variables is defined to be

$$H\{p\} = \int\int p(x,y)\ln p(x,y)\,dx\,dy,$$

where $p(x,y)$ is the probability density function, and the integral is over the set where $p(x,y) > 0$. Note that all probability density functions satisfy the constraint that

$$\int\int p(x,y)\,dx\,dy = 1,$$

because probabilities must always add to one.

Given only the information that the variables (x,y) lie in the unit square $[0,1]\times[0,1]$, derive the maximum entropy distribution.

Solutions: 3 marks We seek to maximum the functional given by entropy, subject only to the constraints that $p(x,y) = 0$ outside the unit square, and $int \int p(x,y)\,dx\,dy = 1$, so

$$\max H\{p\} = \int_0^1 \int_0^1 p(x,y)\ln p(x,y) - \lambda p(x,y)\,dx\,dy,$$

The Euler-Lagrange equation is

$$\frac{\partial f}{\partial p} - \frac{\partial}{\partial x}\frac{\partial f}{\partial p_x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial p_y} = 0$$

Note that f does not depend on the partial derivative p_x or p_y, and so the Euler-Lagrange equation reduces to

$$\frac{\partial f}{\partial p} = \ln p + 1 - \lambda = 0$$

The result is that

$$p(x,y) = const.$$

Given the constraint $int \int p(x,y)\,dx\,dy = 1$ we can see the constant must be 1, so

$$p(x,y) = 1.$$

This is usually called the Uniform Distribution.

4. In Newton's aerodynamical problem we minimized resistance

$$F\{y\} = \int_0^R \frac{x}{1+y'^2}\, dx,$$

subject to $y(0) = L$ and $y(R) = 0$ (and $y' \leq 0$ and $y'' \geq 0$).

In nose-cone design this is sometimes approximated by assuming that the nose-cone will be long and thin, so y' will be large (and negative in our formulation). In that case, we may approximate $1 + y'^2$ by y'^2 and simplify the problem as shown in tutorial 2.

Now, using this approximation, consider an alternative formulation of the problem where we don't specify the length of the nose-cone, we specify the maximum surface area (often called the "wetted area") of the nose-cone. Using the approximation, derive the optimal shape of the nose-cone under this constraints.

Solution: The surface area of a surface of revolution (about the z axis) is given by

$$S\{r\} = 2\pi \int_{r_0}^{r_1} r\sqrt{1+r'^2}\, dz,$$

where r is the radius at height z.

In the co-ordinates of our problem, $r = x$, and $y = z$, so

$$S\{y\} = 2\pi \int_0^L x\sqrt{1+dx/dy^2}\, dy,$$

However, we note that because $y' < 0$ we can invert the function describing the shape of the nose-cone to get

$$S\{y\} = 2\pi \int_0^L x\sqrt{1+y'^2}\, dx,$$

When we apply the large y' approximation this becomes

$$S\{y\} \simeq 2\pi \int_0^L xy'\, dx,$$

and we will constrain this to be equal to S. This is effectively an isoperimetric constraint, so the problem becomes minimize

$$F\{y\} = \int_0^R \frac{x}{y'^2} + \lambda x y'\, dx.$$

Note also that the end-point conditions were loosened so we only require $y(R) = 0$.

Note there is no y term in the integral so the Euler-Lagrange equations are

$$\begin{aligned} \frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} &= 0 \\ \frac{d}{dx}\left[\frac{-2x}{y'^3} + \lambda x\right] &= 0 \\ x\left[\frac{-2}{y'^3} + \lambda\right] &= c_1. \end{aligned}$$

Note that at $x = 0$, the above is zero, but it must be constant for all x, and this means that the constant $c_1 = 0$, and so

$$x\left[\frac{-2}{y'^3} + \lambda\right] = 0,$$

which given $x > 0$ in turn implies that

$$\frac{-2}{y'^3} + \lambda = 0,$$

or

$$y' = const,$$

so

$$y = mx + c.$$

Note that the value of x is fixed at zero, but $y(0)$ is not fixed, so the natural boundary condition is

$$\left.\frac{\partial f}{\partial y'}\right|_{x=0} = x\left.\left[\frac{-2}{y'^3} + \lambda\right]\right|_{x=0} = 0,$$

but this is automatically true given the form of $\frac{\partial f}{\partial y'}$, so we gain no information from this condition.

If the section of the nose-cone is straight, then the nose-cone is a true cone. Then the length of the cone will be determined by the surface area constraint, where we know the exact formula for the surface area of a cone (ignoring the base) is obtained by unwrapping it to get a segment of a circle, so

$$S = \pi R\sqrt{L^2 + R^2},$$

if the cone has length L. From the constraint on length, we obtain $L = y(0)$, and from this and the condition that $y(R) = 0$ we may obtain the two constants m and c.

Note that we should really have added into the formulation the surface area of any flat area at the tip of the nose-cone. However, we already knew that when we take the large y' approximation, there will be no such area, and we see our resulting solution has no such area either.

Extras: If, instead, we had restricted the volume of the nose cone, the solution would be different. The volume of a solid of rotation is

$$V\{r\} = \pi \int_{r_0}^{r_1} r^2\sqrt{1+r'^2}\, dz \simeq \pi \int_0^L x^2 y'\, dx.$$

If we put this into the objective functional with a Lagrange multiplier we get

$$F\{y\} = \int_0^R \frac{x}{y'^2} + \lambda x^2 y'\, dx.$$

with $y(R) = 0$. The Euler-Lagrange equations give

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{d}{dx}\left[\frac{-2x}{y'^3} + \lambda x^2\right] = 0,$$

and again $\frac{\partial f}{\partial y'} = 0$ at $x = 0$ (either because it contains a factor of x or through the natural boundary condition), and so we get

$$\begin{aligned} \frac{-2x}{y'^3} + \lambda x^2 &= 0 \\ y'^3 &= -\lambda x^{-1}/2 \\ y &= c_1 x^{2/3} + c_2, \end{aligned}$$

where the values of the constants again arise from the constraints. Consider the contrast of these three large y' solutions

- Boundary constraints only $x \sim y^{4/3}$
- Surface area constraint $x \sim y$
- Volume constraint $x \sim y^{3/2}$

The family of these solutions $x \sim y^\alpha$ for $\alpha \in [0, 1]$ is called the power-series or sometimes parabaloid nose cone, even though formally only the case $\alpha = 1/2$ is a parabola. Note also that the last of the three isn't convex, and so violates our conditions for the shape.