

Variational Methods & Optimal Control

lecture 28

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Feedback control systems

- ▶ control problem until now have been planned.
 - ▷ know the problem before hand
 - ▷ assume state is perfectly observable
 - ▷ plan the control from the start
- ▶ alternative: feedback control
 - ▷ observe the state at time t
 - ▷ make decisions on the best control at time t
 - ▷ continually update this decision

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Feedback control systems

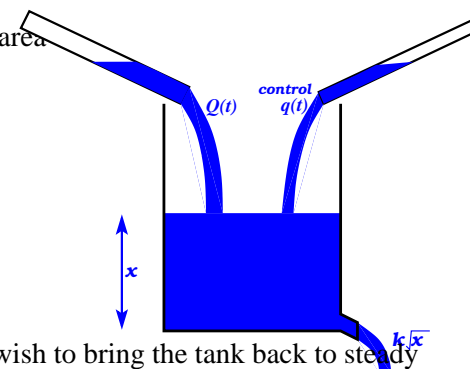
In all of our previous examples, we solve optimization problem “all at once”, i.e., we plan the shape of the curve y to optimize the functional. However, sometimes, we need a control that reacts continuously to perturbations in a system. Such controllers typically utilize feedback.

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Feedback control example

Liquid level control system

- ▶ Tank of uniform cross-sectional area
- ▶ Fluid pumped in at rate $Q(t)$
- ▶ Fluid level $x(t)$
- ▶ Flow out $k\sqrt{x}$
- ▶ Given constant flow, then steady-state will be $x_s = (Q/k)^2$
- ▶ when there are fluctuations, we wish to bring the tank back to steady state, by addition of suitable input at rate $q(t)$.



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Feedback control example

Liquid level optimal control problem

- ▶ we wish to operate near steady state, so part of cost is the square deviation

$$\int_0^T [x(t) - x_s]^2 dt$$

- ▶ Also want to minimize control expenditure, e.g.

$$\int_0^T q(t)^2 dt$$

- ▶ Problem is to minimize a linear combination of these two, e.g.

$$\int_0^T [(x(t) - x_s)^2 + \alpha^2 q(t)^2] dt$$

Feedback control example

So the optimal control problem can be written as minimize

$$F\{y\} = \int_0^T [y(t)^2 + \alpha^2 q(t)^2] dt$$

subject to $\dot{y} = -Ay + q(t)$, and $y(0) = x(0) - x_s$ and $y(T) = 0$.

Substitute $q(t) = Ay + \dot{y}$ into the integral and we get

$$F\{y\} = \int_0^T [y(t)^2 + \alpha^2 (Ay + \dot{y})^2] dt$$

so the E-L equations are

$$\frac{d}{dt} \frac{\partial f}{\partial \dot{y}} - \frac{\partial f}{\partial y} = 2\alpha^2 \frac{d}{dt} [\dot{y} + Ay] - 2y - 2\alpha^2 [A\dot{y} + A^2 y] = 0$$

Feedback control example

The dynamics of the system are

$$\dot{x} = Q + q - k\sqrt{x}$$

We can linearize the problem as follows:

- ▶ note we wish to maintain x near x_s .
- ▶ can approximate x near x_s by a Taylor series

$$\sqrt{x} = \sqrt{x_s} + \frac{1}{2\sqrt{x_s}}(x - x_s) + O((x - x_s)^2)$$

- ▶ the steady state will be when $\dot{x} = 0$, so $Q = k\sqrt{x_s}$
 - ▶ change variables to $y = x - x_s$, so given $A = k/2\sqrt{x_s}$
- $$\dot{y} = -Ay + q(t)$$

Feedback control example

$$2\alpha^2 \frac{d}{dt} [\dot{y} + Ay] - 2y - 2\alpha^2 [A\dot{y} + A^2 y] = 0$$

$$\alpha^2 \frac{d}{dt} [\dot{y} + Ay] - y - \alpha^2 [A\dot{y} + A^2 y] = 0$$

$$\alpha^2 [\ddot{y} + A\dot{y} - A\dot{y} - A^2 y] - y = 0$$

$$\ddot{y} - \left[\frac{1 + \alpha^2 A^2}{\alpha^2} \right] y = 0$$

This has solution

$$y(t) = Ce^{\lambda t} + Be^{-\lambda t}$$

where $\lambda = \sqrt{\frac{1 + \alpha^2 A^2}{\alpha^2}}$.

Feedback control example

The solution is

$$y(t) = Ce^{\lambda t} + Be^{-\lambda t}$$

Given the end-point constraint that $y(0) = y_0$, we get conditions $C + B = y_0$, so the solution is in the form

$$y(t) = y_0 \left(ae^{\lambda t} + (1-a)e^{-\lambda t} \right)$$

Take the case where $T \rightarrow \infty$, and we wish $y(T) = 0$, i.e. the fluctuation should go to zero in the far future, then, we require $a = 0$, and the solution will be

$$y(t) = y_0 e^{-\lambda t}$$

where $\lambda = \sqrt{\frac{1+\alpha^2 A^2}{\alpha^2}}$. We get $q(t)$ from $q(t) = Ay + \dot{y}$ which gives

$$q(t) = (A - \lambda)y_0 e^{-\lambda t} = (A - \lambda)y(t)$$

Feedback control example

Given a proportional controller, it is easy to rewrite the dynamic equation

$$\dot{y} = -Ay + q(t)$$

using $q(y) = My$ as

$$\dot{y} + (A - M)y = 0$$

which has solution

$$y(t) = y_0 e^{-(A-M)t}$$

We wish to choose M so that it minimizes

$$F\{y\} = \int_0^T [y(t)^2 + \alpha^2 q(t)^2] dt$$

Feedback control example

- ▶ the above gives the optimal control policy $q(t)$ over the whole interval $[0, T]$
- ▶ actually, a feedback control problem would be more convenient
 - ▷ given the state $y(t)$, what should the control be
 - ▷ write the control as a function of y , e.g. $q(y)$
- ▶ example: **proportional control**

$$q(y) = My$$

where M is called the **gain**

- ▷ we choose proportional control because in previous planned solution $q(t) = (A - \lambda)y_0 e^{-\lambda t}$ which is proportional to the perturbation $y(t)$

Feedback control example

$$\begin{aligned} F\{y\} &= \int_0^T [y(t)^2 + \alpha^2 q(t)^2] dt \\ &= \int_0^T [y(t)^2 + \alpha^2 M^2 y(t)^2] dt \\ &= (1 + \alpha^2 M^2) \int_0^T y(t)^2 dt \\ &= y_0 (1 + \alpha^2 M^2) \int_0^T e^{-2(A-M)t} dt \\ &= y_0 \frac{(1 + \alpha^2 M^2)}{-2(A-M)} \left[e^{-2(A-M)t} \right]_0^T \end{aligned}$$

Feedback control example

Take the case $T \rightarrow \infty$ and we get

$$F\{y\} = y_0 \frac{(1 + \alpha^2 M^2)}{2(A - M)}$$

In order to find a maximum we differentiate WRT M , to get

$$\begin{aligned} \frac{d}{dM} F\{y\} &= y_0 \frac{\alpha^2 2M 2(A - M) + 2(1 + \alpha^2 M^2)}{4(A - M)^2} \\ &= y_0 \frac{2\alpha^2 M(A - M) + (1 + \alpha^2 M^2)}{2(A - M)^2} \end{aligned}$$

To get a minimum, the numerator must be zero, so we set

$$2\alpha^2 M(A - M) + (1 + \alpha^2 M^2) = 0$$

Feedback control example

$$\begin{aligned} 2\alpha^2 M(A - M) + (1 + \alpha^2 M^2) &= 0 \\ -\alpha^2 M^2 + 2\alpha^2 AM + 1 &= 0 \\ M^2 - 2AM - 1/\alpha^2 &= 0 \end{aligned}$$

This is just a quadratic equation in M , which we solve to get

$$M = \frac{2A \pm \sqrt{4A^2 + 4/\alpha^2}}{2} = A \pm \sqrt{A^2 + 1/\alpha^2}$$

If we take the solution with the '+' sign, then $F\{y\} = y_0 \frac{(1 + \alpha^2 M^2)}{2(A - M)}$ will be negative (the denominator is negative), so we are restricted to

$$M = A - \sqrt{A^2 + 1/\alpha^2}$$

Feedback control example

► $M < 0$, which makes sense, because the control has to reverse fluctuations.

► Compare the feedback and planned solutions

$$q(t) = (A - \lambda)y_0 e^{-\lambda t}$$

$$q(t) = My(t) = \left(A - \sqrt{A^2 + 1/\alpha^2}\right) y_0 e^{-(A - M)t}$$

where $\lambda = \sqrt{\frac{1 + \alpha^2 A^2}{\alpha^2}}$, and notice they are the same.

► for α large, e.g. high control cost, the solution has $1/\alpha^2 \rightarrow 0$, and so $M \rightarrow 0$.

► for α small, e.g. low control cost, the fluctuation cost dominates, and $M \rightarrow -1/\alpha$