Tutorial 6: Wednesday 24th October

1. **Maximize the range of a missile:** Take a missile which has a rocket motor that generates constant thrust f for a fixed time interval $[0, t_1]$. We can control the angle of the thrust $\theta(t)$ (relative to the horizontal). Ignoring drag, the curve of the Earth's surface (and its rotation), determine the angle profile that will maximize the range of the missile.

   Hints: choose a co-ordinates (x, y), and $(u, v) = (\dot{x}, \dot{y})$, then the DEs describing the system under thrust will be

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= f \cos\theta \\ \dot{v} &= f \sin\theta - g \end{aligned}$$

   After the rocket stops firing, the missile will continue on a ballistic trajectory, i.e., the remaining motion will be a parabola, resulting in a total firing distance of

$$R(x, y, u, v) = x + \frac{u}{g}\left[v + \sqrt{v^2 + 2gy}\right]$$

   where x, y, u, v are given at the time at which ballistic motion commences.

2. **Conservation laws:** Consider the simple 2D harmonic oscillator, i.e, an oscillator whose kinetic and potential energies are described by

$$\begin{aligned} T &= \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) \\ V &= \frac{\omega^2}{2}\left(q_1^2 + q_2^2\right). \end{aligned}$$

   (a) Consider whether this system has translation and/or rotational symmetries, and using Noether's theorem describe the conservation laws that apply.

   (b) Now transform the system into co-ordinates

$$\begin{aligned} x_1 &= \frac{1}{2}(q_1 - iq_2) \\ x_2 &= \frac{1}{2}(q_1 + iq_2). \end{aligned}$$

   Show the the resulting system is invariant under the continuous familiy of "squeeze" transforms

$$\begin{aligned} X_1 &= e^{\varepsilon} x_1 \\ X_2 &= e^{-\varepsilon} x_2 \end{aligned}$$

   and derive the corresponding conservation law.

(c) Have we discovered a new conservation law for the system? Explain.

3. Optimal control: Solve the following optimal control problem: find the control $0 \leq u(t) \leq 1$ that minimizes

$$F\{u\} = \int_0^T x_1 u - x_2 u \; dt$$

subject to the system DEs

$$\begin{aligned} \dot{x}_1 &= 1 - u \\ \dot{x}_2 &= x_1 + 1 \end{aligned}$$

Given starting point $(x_1, x_2) = (0, 0)$ at time 0, and end-point $(x_1, x_2) = (1, 2)$ derive the time T at which we reach the end-point.