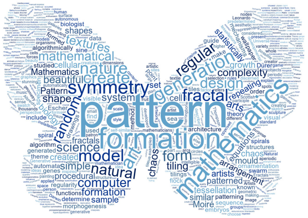
Advanced Mathematical Perspectives 1

Lecture 5: Symmetry and tessellations: the language of patterns



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Section 1

Symmetry

Tyger Tyger, burning bright,
In the forests of the night;
What immortal hand or eye,
Could frame thy fearful symmetry?

The Tyger, William Blake, 1794

Mathematical Symmetry

Symmetry means *invariant* with respect to a particular *transformation,* Thus

- Rotational symmetry means an object or pattern than is invariant with respect to a class of rotations
- Translational symmetry means an object or pattern than is invariant with respect to a class of translations
- Reflectional symmetry means an object or pattern than is invariant with respect to a class of reflections

Symmetries in 2D

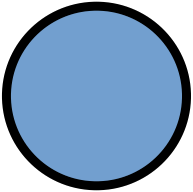
I will concentrate on 2D symmetries:

- rotation
- translation
- reflection
- hybrid, *e.g.*, glide reflection
- scaling

These generalise to higher dimensions (and other types of space), but in higher dimensions there can be additional symmetries.

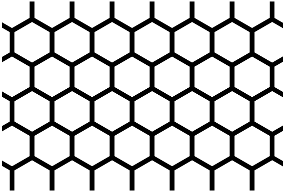
- *e.g.*, helical

Rotational Symmetry



Only a circular pattern (in 2D) is symmetric WRT all rotations.

Starfish (ideally) have 5-fold rotational symmetry (*i.e.*, symmetry under rotations of $360/5 = 72$ degrees).

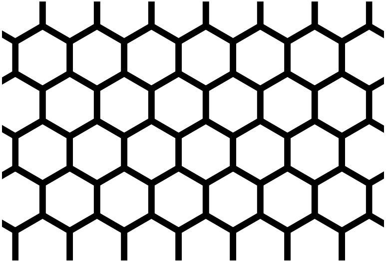


A pattern can have more than one centre of rotation: here centres of hexagons have 6-fold, and vertices have 3-fold rotational symmetry.

Starfish from https://en.wikipedia.org/wiki/File:Fromia_monilis_(Seastar).jpg, by Nhobgood Nick Hobgood.

Translational Symmetry

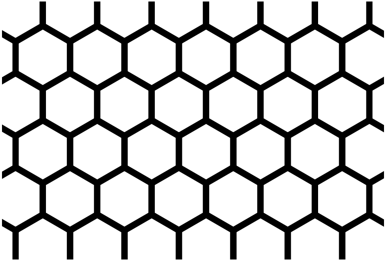
Translational symmetry in x direction.



Translation symmetry in x and y-directions, and diagonally.

Reflectional Symmetry

Bilateral symmetry.

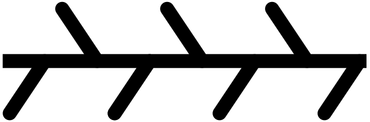


Reflection symmetry in x and y-directions, and diagonally.

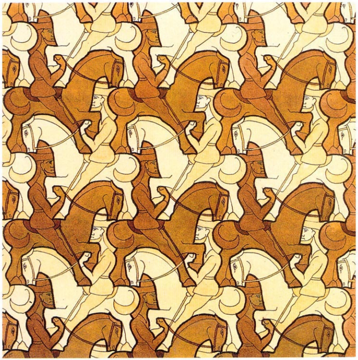
Spider crab from https://en.wikipedia.org/wiki/File:Maja_crispata_(Maia_verrucosa)_-_Museo_Civico_di_Storia_Naturale_Giacomo_Doria_-_Genoa,_Italy_-_DSC03222_Cropped.JPG.

Glide-Reflection Symmetry

We can construct hybrid symmetries as well,
e.g., translation + reflection = glide reflection



Allows translations, and glide-reflections (with half translation distance).

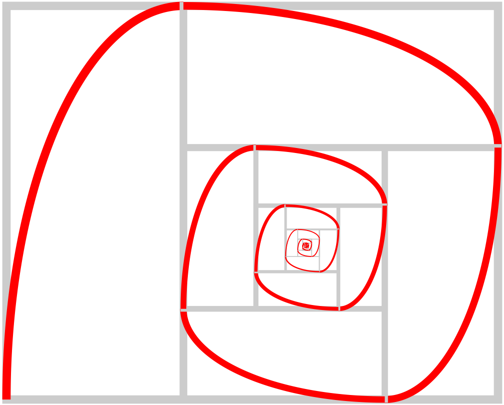


MC Escher's Horsemen.

The point is, it might not have reflection, or (the same) translation but it has the combination.

Scale Symmetry

e.g., Spirals



More on scale later ...

Symmetry Groups

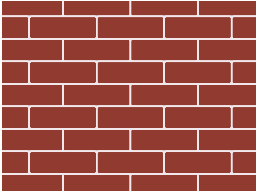
When we get into the math of symmetry there is much more

- symmetry can be used in a more general sense about operations in maths
- we can start to form "sets" of symmetry operations, and then these form a mathematical "group"
  - these become important in crystallography and quantum physics
  - for a flavour of this, lets look at the "wallpaper" group

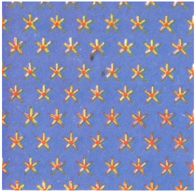
Wallpaper Symmetry Classification

One way to classify patterns is by their symmetries

- Wallpaper symmetry: patterns that
  - are in the 2D Euclidean plane, and extend indefinitely
  - have two independent translational symmetries
  - have discrete "cells"
- There are only 17 possible cases, *e.g.*, 3 listed below



A typical suburban brick wall has 2 perpendicular reflection axes and 2-fold rotation (centre not on reflection axes) and two rotations (centres on reflection axes).



Dado from Biban el Moluk, Egypt. Has reflections (parallel axes), at least one glide reflection (which is not on a reflection axis), no rotations.

Herringbone pattern has glide reflections only, with parallel axes. No rotations or reflections.

Figures from https://en.wikipedia.org/wiki/Wallpaper_group

Section 2

Tessellation

The laws of the phenomena around us–order, regularity, cyclical repetition, and renewals–have assumed greater and greater importance for me. The awareness of their presence gives me peace and provides me with support. I try in my prints to testify that we live in a beautiful and orderly world, and not in a formless chaos, as it sometimes seems.

M. C. Escher

Symmetries in Patterns

Lots of main-made patterns contain symmetries

- *e.g.*, actual wallpaper
- *e.g.*, bricks and pavers, *e.g.*, `http://www.geckostone.com/pavers.html`
- *e.g.*, textiles (carpets, clothes, quilts, upholstery, ...)
- ...

Also appear in nature

- bilateral symmetry in animals
- rotational symmetry, *e.g.*, starfish

But for the moment we are more interested in repeating patterns

- *e.g.*, crystals
- *e.g.*, honeycomb

Definition of Tessellation

Let's (mostly) stay on the 2D Euclidean plane $\mathbb{R}^2$. Roughly, a tessellation[1] is where we cover said plane using one or more geometric shapes called *tiles*, with no overlaps or gaps. More formally:

Definition

A tesselation or tiling of the plane is a *countable* collection of *closed* sets (the tiles), intersecting only on their boundaries, whose union *covers* the plane.

- There are many specialised tilings, for instance, typically the tiles are expected to be topologically disks (they have no holes), and are uniformly bounded – this is called a *normal* tessellation.
- The definition is very general, so there are a few other pieces of terminology worth defining alongside it.

[1] The word tessellation comes from the latin *tessella*, a small (square) piece of a mosaic. So the terms tiling and tessellation are directly linked

Terminology

Definition

An *edge* is the intersection between two bordering tiles (often a straight line).

Definition

A *vertex* is the point of intersection of three or more tiles. Vertices are the "corners".

Often we insist that the vertices cannot lie on the edge of another tile, *i.e.*,

Definition

A polygonal tiling is *edge-to-edge* if adjacent tiles share exactly one full side.

The "brick wall" tiling violates this criteria.

Terminology

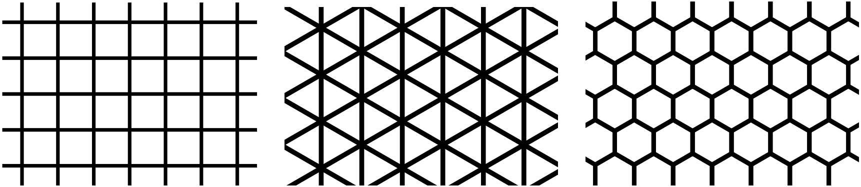
Most of the math around concerns other specific, restricted tilings.

- *isogonal*: the every vertex is identical.
- *regular*: the tiles are identical and have identical vertices. This effectively limits them to be regular polygons because all of the angles must be the same.
- *semi-regular*: the tiles can use more than one type of regular polygon, but the arrangement of polygons at each vertex is identical.
- *periodic*: the pattern repeats, *i.e.*, the pattern has (at least) two translational symmetries.
  These all fall into the 17 wallpaper groups.
  Regular and semi-regular tilings are all periodic, but there are other periodic tilings (there are only 3 regular, and 8 semiregular tilings, but 17 wallpaper groups, so we know we missed at least some).
- *monohedral*: a tiling in which all tiles are *congruent* (the same shape, possibly rotated and reflected).

Regular Tilings

Regular tilings are constructed from regular polygons

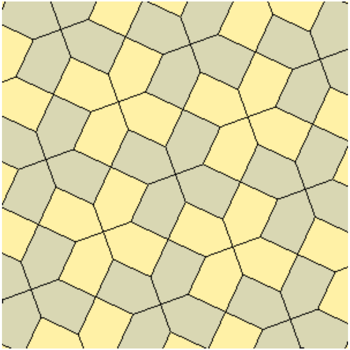
- Cases: there are only 3



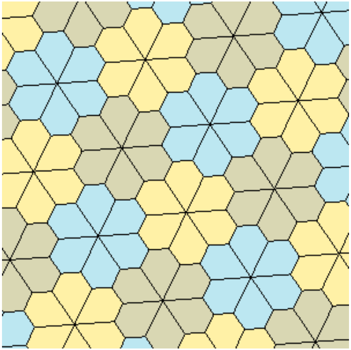
- Notation: choose a vertex, label by the no. of sides for each polygon
  - above we have: 4,4,4,4 and 3,3,3,3,3,3 and 6,6,6
  - or just $\{4,4\}$ and $\{3,6\}$ and $\{6,3\}$

Not Quite Regular Tilings

- Note that some simple regular polygons can't form tilings, the most obvious case being the pentagon
- But we can form monohedral tilings with irregular pentagons



https://en.wikipedia.org/wiki/File:Pentagonal_tiling_type_4_animation.gif



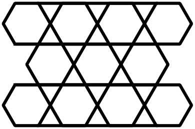
https://en.wikipedia.org/wiki/File:Pentagonal_tiling_type_5_animation.gif

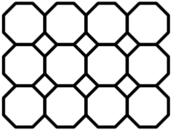
- There are actually quite a lot of possibilities [Rho05]

Semi-regular tilings

Semi-regular tesselations can have more than one type of (regular) polygon, but the arrangement at each vertex is identical.

- There are 8 semi-regular tesselations


3,6,3,6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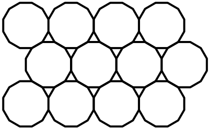
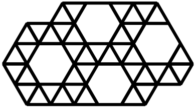
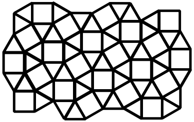
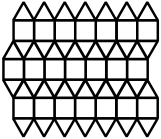

3,12,12


3,3,3,3,6


3,3,4,3,4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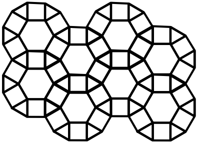
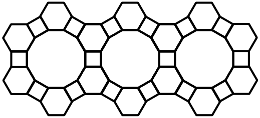

3,4,6,4


4,6,12

Semi-regular tilings

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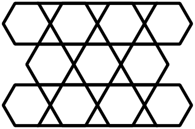
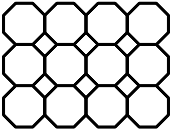
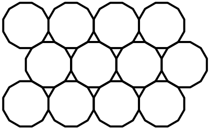
- There are 8 semi-regular tesselations



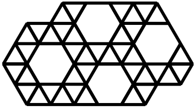
3,6,3,6



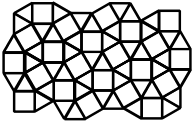
4,8,8



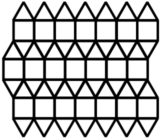
3,12,12



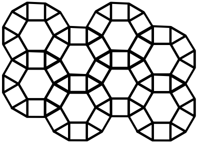
3,3,3,3,6



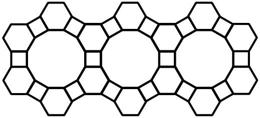
3,3,4,3,4



3,3,3,4,4



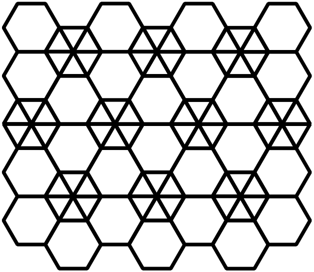
3,4,6,4

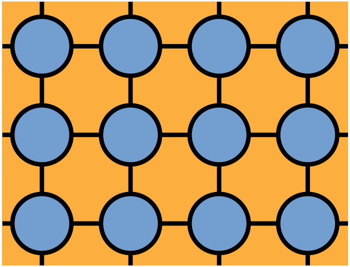


4,6,12

Other not-quite-so regular tilings

- Some seem like they should work, but can't be extended infinitely (see practical)
- Lots of other possibilities still using regular polygons if we loosen up the vertices all being the same, *e.g.*, k-uniform tilings
- Lots of possibilities for non-edge-to-edge tilings
- Nothing here says you can't have curved shapes in tessellation, and that opens up lots of possibilities, *e.g.*,





Make your own

- https://fashion-incubator.com/pattern-puzzle-escher-cad-fun-for-all/
- http://www.geckostone.com/pavers.html
- http://www.computersforcreativity.com/resources/inkscape/inkscapetessallations
- Note Inkscape has a tool for making polygons, and if we duplicate, and shift these, we can build tesselations. I will provide some examples.
- http://verysimpledesigns.com/vectors/inkscape-tutorial-seamless-patterns.html
- http://danceswithferrets.org/geekblog/?p=154

But we will have a go in Matlab.

Further reading I

- Jinny Beyer, *Designing tessellations: The secrets of interlocking patterns*, Contemporary Books, 1999.
- John H. Conway, Heidi Burgiel, and Chaim Goodman-Strauss (eds.), *The symmetries of things*, CRC Press, 2008.
- Taylor Collins, Aaron Reaves, Tommy Naugle, and Gerard Williams, *A classification of frieze patterns*, https://www.math.lsu.edu/system/files/RP2Presentation.pdf, July 2010.
- Frank A. Farris, *Creating symmetry: The artful mathematics of wallpaper patters*, Princeton University Press, 2015.
- Glenn C. Rhoads, *Planar tilings by polyominoes, polyhexes, and polyiamonds*, Journal of Computational and Applied Mathematics **174** (2005), no. 2, 329–353, https://www.sciencedirect.com/science/article/pii/S0377042704002195.

Further reading II

Dale Seymour and Jill Britton, *Introduction to tessellations*, Dale Seymour Publications, 1989.