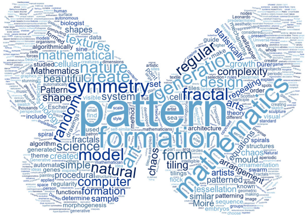
Advanced Mathematical Perspectives 1

Lecture 9: Sinusoids in 2D



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Section 1

Sinusoids In 2D

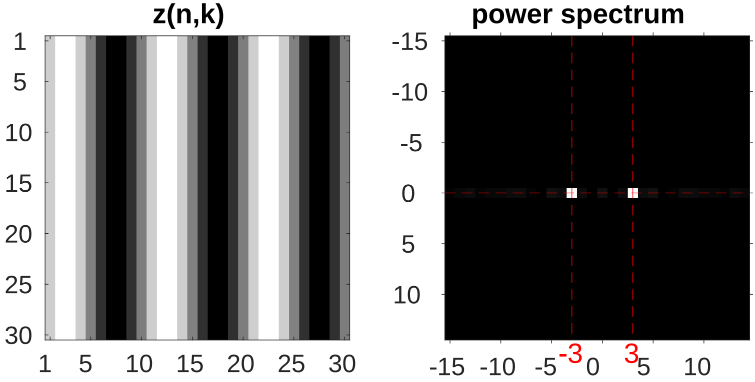
- When we look at a sine wave in 2D it has a *direction*, which we can think of as a vector **T**
  - **T** gives a vector along which the signal is translation invariant (with some period)
  - We can then derive frequencies in the x and y directions
- Let's write the "signal" $z(n, k)$ where
  - we could think of z as a the height of a surface above the x–y plane, or as an image intensity or brightness
  - we are using discrete indices, because we will look at these as raster images, so we can see z as a matrix

$$z(n, k) = z_{n,k}$$

  - the indices n and k will be in the standard matrix sense, *i.e.*,
    - ⋆ k is along the row
    - ⋆ n is along the column

Example (i)

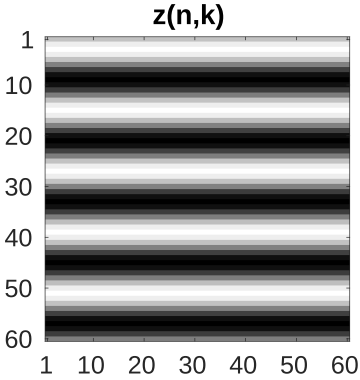
$z(n,k) = \sin\left(2\pi \mathbf{3} k / M\right)$

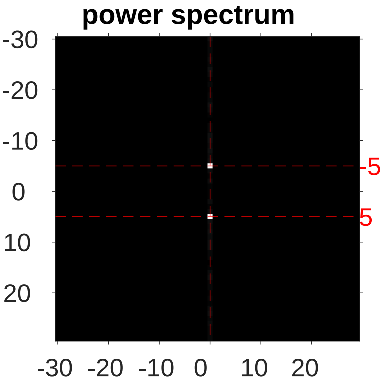


Here $n, k = 1, 2, \ldots, M$ where $M = 30$, and the period $= 10 = M/3$

Example (ii)

$z(n,k) = \sin(2\pi \mathbf{5} n / M)$

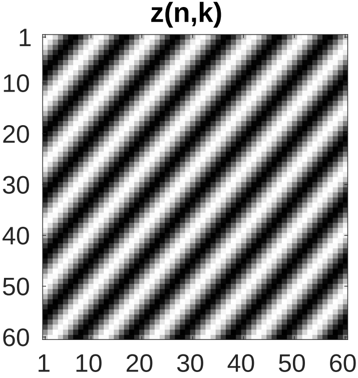


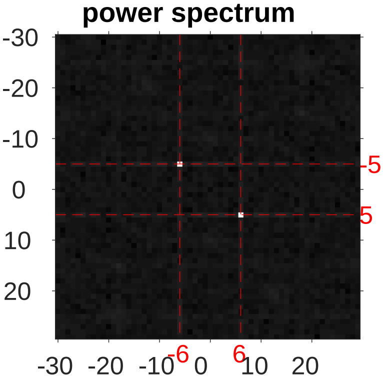


Here $n, k = 1, 2, \ldots, M$ where $M = 60$, and the period $= 12 = M/5$

Example (iii)

$z(n,k) = \sin\left(2\pi[\mathbf{6}k + 5n]/M\right)$



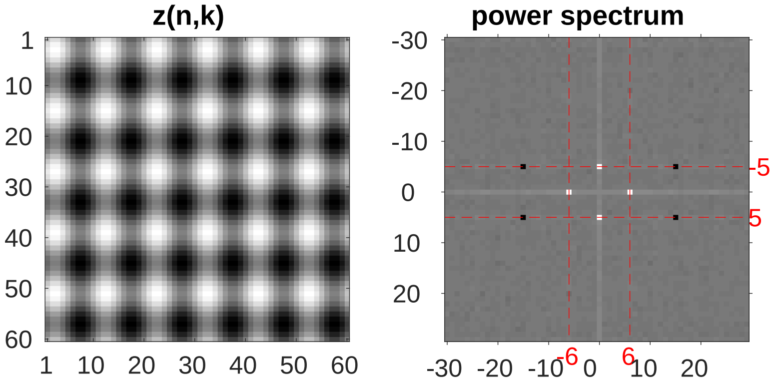


Here $n, k = 1, 2, \ldots, M$ where $M = 60$,
and the period $= (60/6, 60/5) = (10, 12)$

actually the period is half this ...

Example (iv)

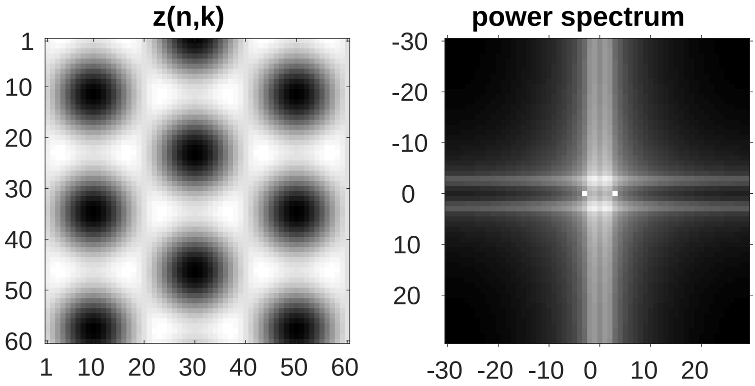
$$z(n, k) = \sin(2\pi \mathbf{6} k/M) + \sin(2\pi \mathbf{5} n/M)$$



Here $n, k = 1, 2, \ldots, M$ where $M = 60$

Example (v)

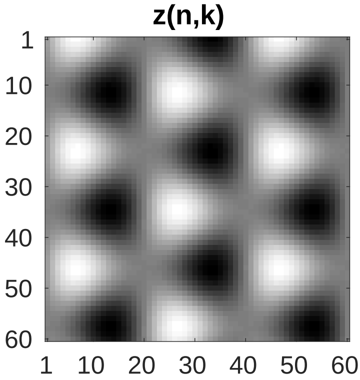
$$z(n,k) =$$
$$\sin\left(\frac{2\pi}{M} f_0 k + \frac{\pi}{2}\right) + \sin\left(\frac{2\pi}{M} f_0 \left[\frac{1}{2} n + \frac{\sqrt{3}}{2} k\right]\right) + \sin\left(\frac{2\pi}{M} f_0 \left[\frac{1}{2} n - \frac{\sqrt{3}}{2} k\right]\right)$$

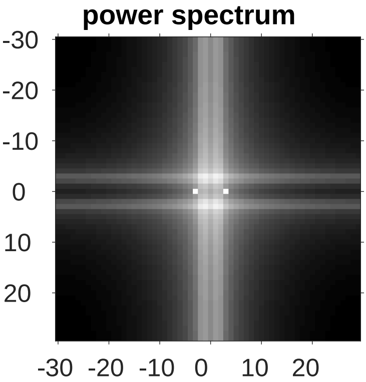


Here $n, k = 1, 2, \ldots, M$ where $M = 60$

Example (vi)

$$z(n,k) = \sin\left(\frac{2\pi}{M} f_0 k\right) + \sin\left(\frac{2\pi}{M} f_0 \left[\frac{1}{2}n + \frac{\sqrt{3}}{2}k\right]\right) + \sin\left(\frac{2\pi}{M} f_0 \left[\frac{1}{2}n - \frac{\sqrt{3}}{2}k\right]\right)$$





Here $n, k = 1, 2, \ldots, M$ where $M = 60$

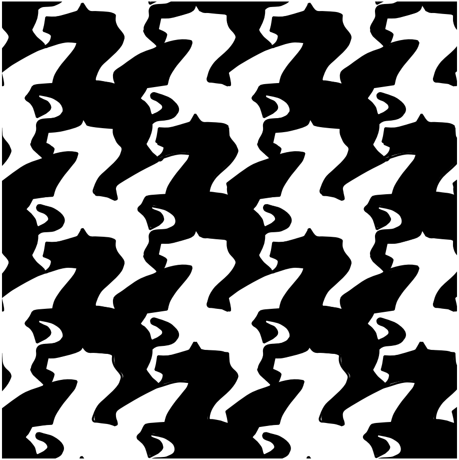
Periodic patterns

- These are all smooth
- But I can generate *any* periodic pattern with a series of sinusoids
  - This holds for 2D patterns just as for 1D
  - I might need quite a lot of them!!!
  - Fourier transform can tell me which patterns are needed

Escher's Pegasus

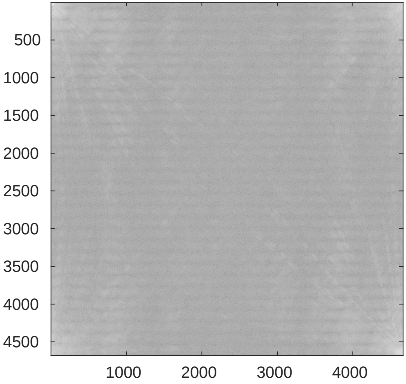
Original

Escher's Pegasus



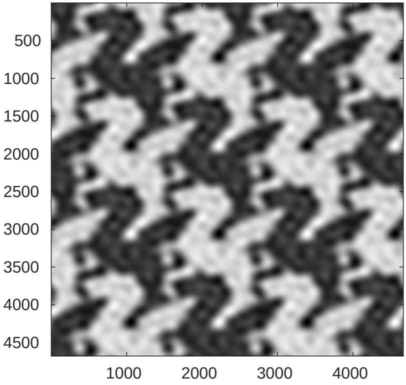
Stylised version

Escher's Pegasus



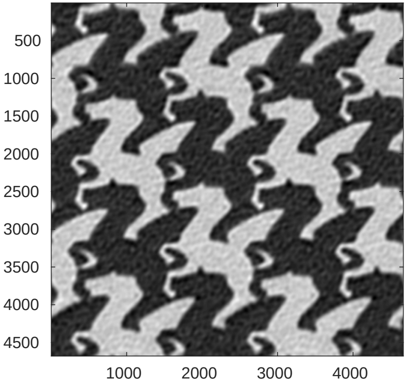
Power spectrum

Escher's Pegasus



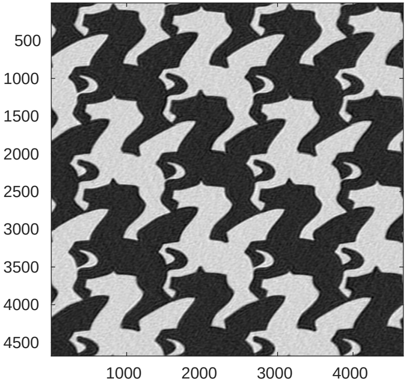
Reconstruction (from 100 sinusoids)

Escher's Pegasus



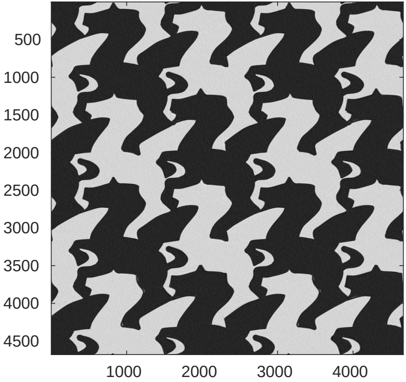
Reconstruction (from 1000 sinusoids)

Escher's Pegasus



Reconstruction (from 10,000 sinusoids)

Escher's Pegasus

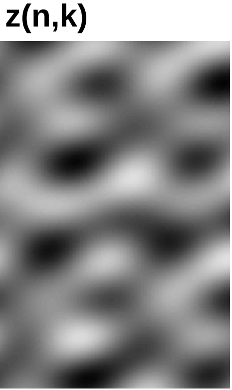


Reconstruction (from 100,000 sinusoids)

Can we generate more interesting patterns?

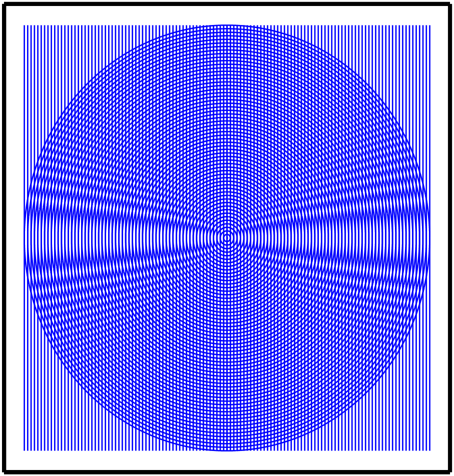
Can we generate more interesting patterns?

Only 5 sinusoids together can generate something that looks random(ish)



Moiré patterns

A phenomena called aliasing causes patterns to emerge from "sampling" of high frequency signals. In 2D this can often be seen when patterns of lines overlap, creating Moiré patterns.



Takeaways

- Periodicity and repetition
  - can construct with sines and cosines
  - can analyse these with Fourier transforms
- Randomness can appear out of determinism

Further reading I