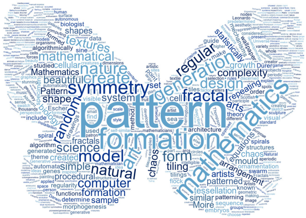
Advanced Mathematical Perspectives 1

Lecture 13: Random Walks



Matthew Roughan

<matthew.roughan@adelaide.edu.au>

www.maths.adelaide.edu.au/matthew.roughan/notes/AMP1/

School of Mathematical Sciences,
University of Adelaide

THE UNIVERSITY of ADELAIDE

SUB CRUCE LUMEN

ACEMS

AUSTRALIAN RESEARCH COUNCIL CENTRE OF EXCELLENCE FOR MATHEMATICAL AND STATISTICAL FRONTIERS

Probability is the very guide of life.

Leonard Mlodinow, The Drunkard's Walk: How Randomness Rules Our Lives

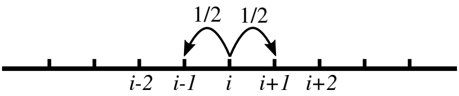
It takes only one drink to get me drunk. The trouble is, I can't remember if it's the thirteenth or the fourteenth.

George Burns

Section 1

The Drunkard's Walk

A Simple Random Walk



- Imagine moving around the integers using the following rule:
  - starting at 0
  - at each time step, toss a fair coin
  - if heads, move left by 1
  - if tails, move right by 1

  This is a simple *random walk*
- Questions:
  - *If we did this lots of times, what is the probability distribution across the set of points?*
  - If we had barriers, how long would it take to hit them?
  - How often does the walk cross over itself (particularly in 2D)?
  - ...

Random Walk Generalisations

We can generalise this in various ways

- allow a biased coin
- allow larger jumps
- jump around on a 2D lattice of points

But for the moment let's keep it simple

Random Walk Mathematics

Take a series of *random variables* $\{X_i\}$ for $i = 1, 2, \ldots$ defined by

$$X_i = \begin{cases} 1, & \text{with probability } 1/2, \\ -1, & \text{with probability } 1/2. \end{cases}$$

Now we could describe the state of our random walk at time n as a random variable S_n, defined by $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} X_i$$

This is a very common type of random process, and often analysed.

Bernoulli distribution

A *Bernoulli* random variable is X such that

$$X_i = \begin{cases} 0, & \text{with probability } 1-p, \\ 1, & \text{with probability } p. \end{cases}$$

Think of it as flipping a biased coin, with probability p of heads (or a "success")

Binomial distribution

The *Binomial* random variable $Y \sim B(n, p)$ is what we get when we sum n Bernoulli random variables. It has probability distribution:

$$\text{Prob}(Y = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

It has mean $\mathbb{E}[Y] = np$ and variance $\text{Var}(Y) = np(1-p)$.

The random-walk random variable is almost the same as the Binomial, except the components take values -1 and 1 instead of 0 and 1.

Random Walk Distribution

- The random-walk random variable is almost the same as the Binomial, except the components take values -1 and 1 instead of 0 and 1.
- The main difference is
  - at time 1, we can only be in state -1 or 1
  - at time 2, we can only be in state -2, 0, or 2

  so at odd times = odd state, and even times = even state
- So, combining insight from Binomial, and the above we get a distribution following Pascal's triangle, *i.e.*, $\binom{n}{k}$

k	-4	-3	-2	-1	0	1	2	3	4
$\text{Prob}\left(S_n = 0\right)$					1				
$2\text{Prob}\left(S_n = 1\right)$				1		1			
$2^2\text{Prob}\left(S_n = 2\right)$			1		2		1		
$2^3\text{Prob}\left(S_n = 3\right)$		1		3		3		1	
$2^4\text{Prob}\left(S_n = 4\right)$	1		4		6		4		1

Tricks for understanding Random Walk Distribution

Expectation is the average or mean

Expectation is a *linear operator* which means

$$\mathbb{E}[A + B] = \mathbb{E}[A] + \mathbb{E}[B]$$

And so

$$\mathbb{E}[S_n] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i] = 0$$

Tricks for understanding Random Walk Distribution

Variance is a measure of the spread of variability

Variance of *independent* random variables add

$$\mathrm{Var}\,(A + B) = \mathrm{Var}\,(A) + \mathrm{Var}\,(B)$$

And so

$$\mathrm{Var}\,(S_n) = \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}\,(X_i) = \sum_{i=1}^{n} 1 = n$$

Tricks for understanding Random Walk Distribution

The *Central Limit Theorem* says that for any sum like this[1]

$$\sqrt{n}\left[\frac{S_n}{n} - \mu\right] \to N(0, \sigma^2),$$

where $N(0, \sigma^2)$ denotes the normal or Gaussian distribution with mean 0 and variance σ^2 (where this is the variance of the X_i).

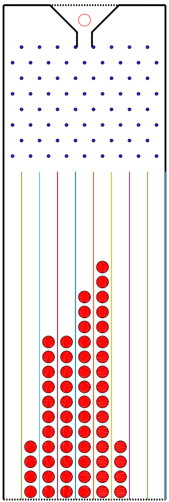
Here, X_i has $\mu = 0$ and $\sigma^2 = 1$, so

$$\frac{S_n}{\sqrt{n}} \to N(0, 1),$$

[1] There are some conditions, and we need to define the notion of limit for probabilities more carefully.

Galton Board

The Galton board or Quincunx demonstrates the Central Limit Theorem



Tricks for understanding Random Walk Distribution

These three together (or just the last) give us a pretty clear picture that the distribution evolves towards a Gaussian (normal) distribution with well understood parameters.

Example

Activities

- Simulate and play with random walks
- Start thinking more seriously about your project

Further reading I

Paul C. Bressloff, *Stochastic processes in cell biology*, ch. Diffusion in Cells: Random Walks and Brownian Motion, Springer, 2014, http://www.springer.com/gp/book/9783319084879.

Sheldon Ross, *Introduction to probability models*, Academic Press, 2010.