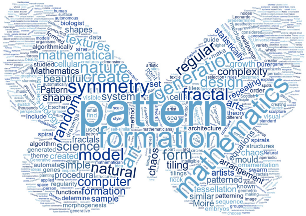
Advanced Mathematical Perspectives 1

Lecture 15: Limits of Random Walks



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Reminder: Random Walk Definition

Take a series of *random variables* $\{X_i\}$ for $i = 1, 2, \ldots$ defined by

$$X_i = \begin{cases} 1, & \text{with probability } 1/2, \\ -1, & \text{with probability } 1/2. \end{cases}$$

Now we could describe the state of our random walk at time n as a random variable S_n, defined by $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} X_i$$

This is a very common type of random process, and often analysed.

Reminder: Limiting Distributions

The *Central Limit Theorem* says that for any sum like this[1]

$$\sqrt{n}\left[\frac{S_n}{n} - \mu\right] \to N(0, \sigma^2),$$

where $N(0, \sigma^2)$ denotes the normal or Gaussian distribution with mean 0 and variance σ^2 (where this is the variance of the X_i).

Here, X_i has $\mu = 0$ and $\sigma^2 = 1$, so

$$\frac{S_n}{\sqrt{n}} \to N(0, 1),$$

[1] There are some conditions, and we need to define the notion of limit for probabilities more carefully.

Section 1

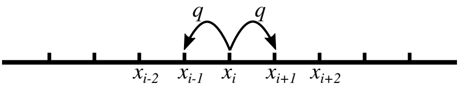
Random Walks and Diffusion

Imagine that instead of a random walk on the integers, we do a random walk on points dx apart, so $x_i = i \times dx$

And we don't jump at every step, so now take

$$X_i = \begin{cases} dx, & \text{with probability } q, \\ 0, & \text{with probability } 1-2q, \\ -dx, & \text{with probability } q. \end{cases}$$

and $S_n = \sum_{i=1}^{n} X_i$



Now represent the probability of being at point x_i at time t_j by

$$\text{Prob}\left(S_j = x_i\right) = p(x_i, t_j)$$

A Random Walk is a Markov Chain

- A *Markov chain* is a random process where we transition (at random) between a set of *states*, and where the next transition depends ONLY on the current state.
  - The Markov property is often called the *memoryless* property because after each transition we "forget" the history of the process
- A random walk (as described above) is a Markov chain
  - the next state depends only on the current state
  - There is lots of theory to learn about Markov chains
    - ★ we touch on them in a few course, but you won't see the real details until Applied Probability in 3rd year
  - But I can do a little bit here, but I won't try to use too much theory that you don't already know (I hope)

A Random Walk is a Markov Chain

The defining property of a Markov chain – lack of memory – can be expressed through conditional probabilities

$$\text{Prob}\left(S_{j+1} = x_i \mid S_0 = x_{k_0}, \ldots, S_j = x_k\right) = \text{Prob}\left(S_{j+1} = x_i \mid S_j = x_k\right)$$

That is, the probability of being in a particular state, given the entire history of the process is equal to the probability given only the most recent state.

For our random walk, we can only jump one step at a time, and so

$$\text{Prob}\left(S_{j+1} = x_i \mid S_j = x_k\right) = \begin{cases} q, & x_i = x_k - dx \\ 1 - 2q, & x_i = x_k \\ q, & x_i = x_k + dx \\ 0, & \text{otherwise.} \end{cases}$$

A Random Walk is a Markov Chain

The Law of Total Probability states[2]

$$p(x_i, t_{j+1}) = \sum_{x_k} \text{Prob}\left(S_{j+1} = x_i \mid S_j = x_k\right) \text{Prob}\left(S_j = x_k\right)$$

Summing over the possible states (at time $n-1$) we get

$$\begin{aligned} p(x_i, t_{j+1}) &= (1-2q)p(x_i, t_j) + qp(x_{i+1}, t_j) + qp(x_{i-1}, t_j) \\ &= p(x_i, t_j) + q\left[p(x_{i+1}, t_j) - 2p(x_i, t_j) + p(x_{i-1}, t_j)\right] \end{aligned}$$

[2]Remember that $p(x_i, t_{j+1}) = \text{Prob}\left(S_{j+1} = x_i\right)$.

Random Walk = Diffusion

So, at each time step, the probability we will be at point x_i will be given by the difference equation

$$p(x_i, t_{j+1}) = p(x_i, t_j) + q\big[p(x_{i+1}, t_j) - 2p(x_i, t_j) + p(x_{i-1}, t_j)\big]$$

Remember the difference equation we found for approximately solving the diffusion equation $\frac{\partial u}{\partial t} = \alpha \nabla^2 u$ was

$$u(x_i, t_{j+1}) = u(x_i, t_j) + \frac{\alpha dt}{dx^2}\big[u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)\big]$$

Note that if we set $q = \alpha dt/dx^2$, the two are identical!

Random Walk = Diffusion

- If we take $q = \alpha dt/dx^2$ the probability equations for our random walk are the same as those for the finite difference approximation to the diffusion equation.
- If we take the limit $dx \to 0$, and $dt \to 0$, such that q is kept constant, the limit of the random walk would be a diffusion process exactly.
  - so we can think of the state probabilities of a random walk with very small steps, in very small time intervals as a diffusion process, *i.e.*, the probabilities can be modelled as "diffusing"
- Underlying this is the nature of the diffusion — a gas is made up of many particles, each bouncing around at random, and so we might think of diffusion as an "average" of all of these random walks.

Random Walk = Diffusion

One of the consequences of this correspondence is that theorems derived in one domain can give me insights into behaviour in the other:

- The Central Limit Theorem tells me that a diffusion will tend towards a Gaussian (normal) distribution (given no boundaries ...)
- The random walk formulation assumed we had no boundaries – it might be useful to model a system with complicated boundaries using the heat equation?

In other ways the two systems vary:

- Probabilities *must* always add to one, but heat can be added or lost from a system.

Section 2

Where are we going?

To Random Walk or Diffuse

So, we can think of

- a random walk as a way of approximating diffusion
- diffusion as a way of approximating a random walk

Which is correct?

Model Selection

- Both *models* are just models
- Neither is "real"
- The model choice that is the best approximation to reality depends on the particular problem
  - Model of heat: *diffusion*; because (i) the size of atoms is so small that we can approximate a solid as a continuum, (ii) we only need concern ourselves with the "average" behaviour, and (iii) diffusion is in some ways simpler.
  - Model of a bank queue: a *random walk* like process, because (i) the number of people is very obviously discrete, and (ii) the random walk gives a better idea of short-term variability, *i.e.*, we don't just care about the average number of people in a queue.
  - Model of disease spread: could be either, depending on the setting and parameters.

Are there other models for this type of thing?

	Discrete	Continuous
Random (Stochastic)	Random Walk	
Deterministic		

Table: Model classes

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	Discrete	Continuous
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Table: Model classes

Are there other models for this type of thing?

	Discrete	Continuous	
Random (Stochastic)	Random Walk	???	models paths
Deterministic	Difference approximation	Diffusion	models averages

Table: Model classes.

Modelling is an Art

- There is no “perfect” model
- Deriving good models requires
  - a big tool box – lots of knowledge about mathematics
  - knowledge of the “physics” of the system being modelled
  - understanding of the trade-offs in various approximations
  - a keen sense of mathematical aesthetics
- I’m trying to give you pieces of this, but particularly the third and forth points

Takeaways

- Random walks!
- There are many ways to model the same system
- Models have various pros and cons

Section 3

Extras

Brownian Motion

- The classic case of *Brownian motion* arises when you observe pollen particles floating in coffee.
  - the particles bounce around follow what we might approximate as a random walk
- But,
  - the length of the jumps can vary
  - the process (the location of the particles) is a continuous state, continuous time process
- Common mathematical model is the Wiener process

Wiener process

- In your workshop, we simulated $1/f$ noise
- The sequence had finite length N, but we could have (in theory) extended this to $N \to \infty$, presuming the gaps between samples $\downarrow 0$
- If $\alpha = 2.0$ this is called the *Wiener process* or sometimes the Brownian motion process
  https://en.wikipedia.org/wiki/Wiener_process
- The process is *continuous* but *stochastic*
- The process is nowhere differentiable, and has lots of other interesting mathematical properties
- Its also *statistically self-similar*

Wiener process as a limit of a random walk

Given a random walk $S_n = \sum_{i=1}^{n} X_i$, take the process

$$W_n(t) = \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}$$

As $n \to \infty$ this is as if we made the intervals between the lattice point go to zero as we did before, but we are looking at the sample path, not the probabilities.

The sample paths form a Wiener process for $t \in [0, 1]$

Brownian Bridge

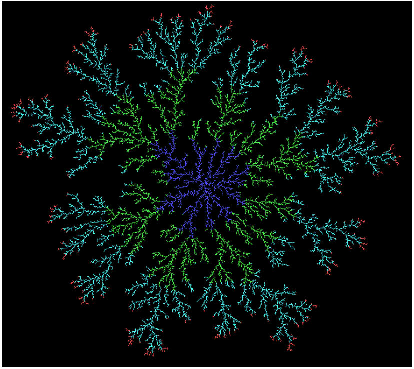
Imagine a Brownian motion (or Wiener process) pinned at both ends.

Technique to generate

- Take a line, and perturb the mid-point by a Gaussian random variable
- Do the same to each of the resulting lines (but scaling the perturbation by $1/\sqrt{2}$)
- Continue this indefinitely

Diffusion Limited Aggregate (DLA)

- Particles follow a random walk (due to Brownian motion), coming to rest when they touch the aggregate
- They form fractal "trees"



https://en.wikipedia.org/wiki/Diffusion-limited_aggregation

Further reading I

Paul C. Bressloff, *Stochastic processes in cell biology*, ch. Diffusion in Cells: Random Walks and Brownian Motion, Springer, 2014, http://www.springer.com/gp/book/9783319084879.

Sheldon Ross, *Introduction to probability models*, Academic Press, 2010.