Complex-Network Modelling and Inference

Lecture 4: Graph connectivity and traversal

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Section 1

Connectivity

Connectivity

Definition

Two nodes are *connected* if a path exists between them.

Definition

A graph is *connected* if all pairs of nodes are connected.

Definition

A *strongly connected* digraph is connected in the sense above, whereas a *weakly connected* digraph is connected if we include all reverse links.

Definition

A graph is *k-edge connected* if the graph remains connected after the removal of any set of $k-1$ edges, and *k-node connected* if the graph remains connected after the removal of any set of $k-1$ nodes.

Definition (Cut)

A *cut* is a partition of the nodes of a graph into two subsets $C = (S, T)$.

Definition (Cut-set)

The *cut-set* of a cut $C = (S, T)$ is the set of edges

$$\{(u, v) \in E | u \in S, v \in T\}$$

i.e., the edges that cross the cut.

Definition (Edge Cut)

An (minimum) *edge cut* is the minimum number of edges whose removal disconnects two nodes i and j, *i.e.*, a minimal cut-set with $i \in S$ and $j \in T$.

Menger's theorem

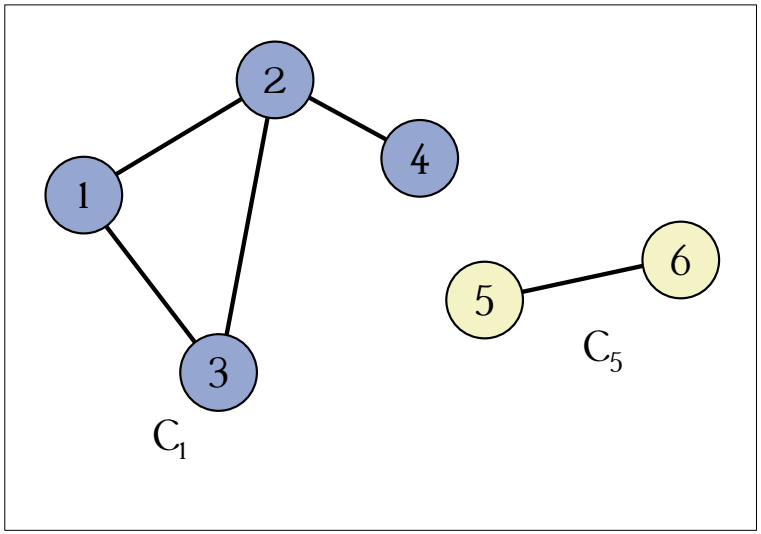
Theorem (Edge-connectivity version)

For an undirected graph G, the size of the minimum edge cut for an arbitrary pair of nodes $i \neq j$ is equal to the maximum number of edge-disjoint paths from i to j.

- Edge-disjoint means they share no common edges
- There is also a node connectivity version
- It also works for digraphs and infinite graphs
- The theorem is generalised in many optimisation algorithms: *e.g.*, maximum flow algorithms.

Connected Components

- A *connected component* is a maximal connected subgraph



- The set of connected components $\{C_i\}$ form a *partition* of the nodes

Definition (Partition)

A *partition* is a set of covering and disjoint subsets, i.e, $\{C_i\}_{i=1}^{n}$ is a partition of C iff

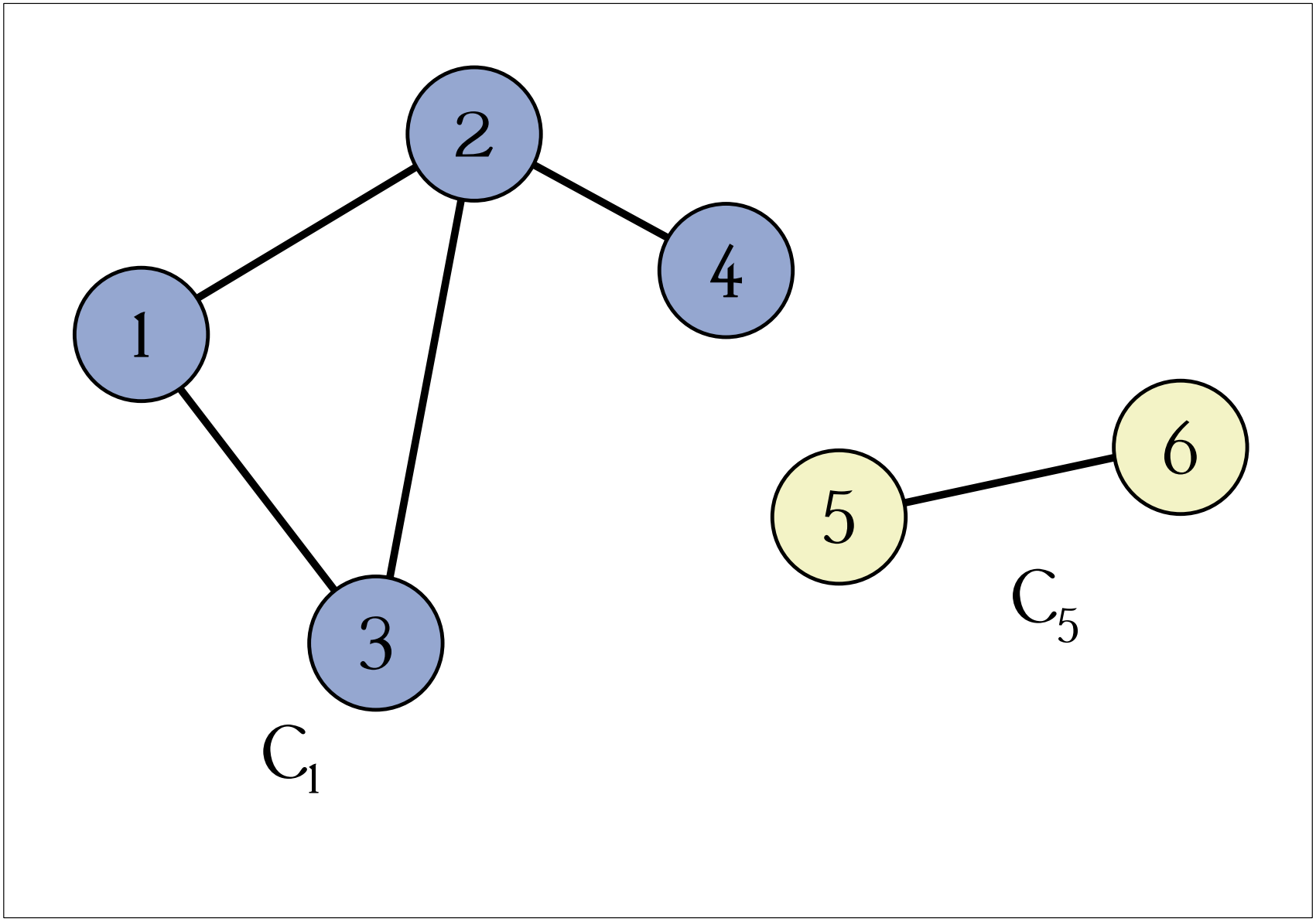
$$\bigcup_{i=1}^{n} C_i = C \quad \text{and} \quad C_i \cap C_j = \phi, \ \ \forall i \neq j.$$

Connected Components Algorithm

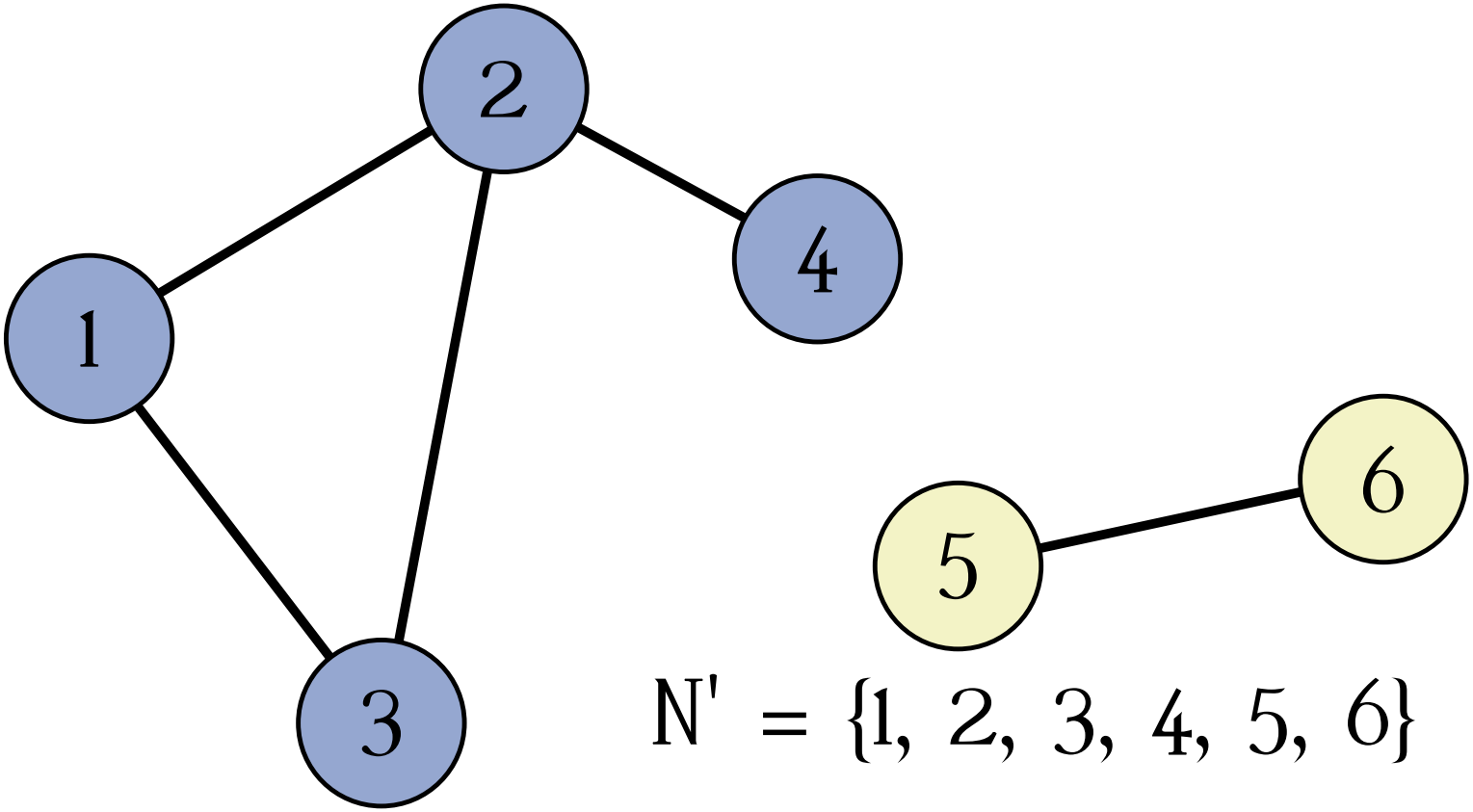
Data: A Graph $G = (N, E)$
Result: A set of connected components $\{C_i\}$

```
1  Initialise N' = N;
2  while (N' ≠ φ) do
3      Choose a node i ∈ N' and delete it from N';
4      Set C_i = {i} and L = {(i,j)|(i,j) ∈ E};
5      while (L ≠ φ) do
6          Choose a link (k,m) ∈ L;
7          if m ∉ C_i then
8              add m to C_i;
9              delete m from N';
10             add all links (m,l) ∈ E to L;
11         end
12         delete (k,m) from L;
13     end
14 end
```

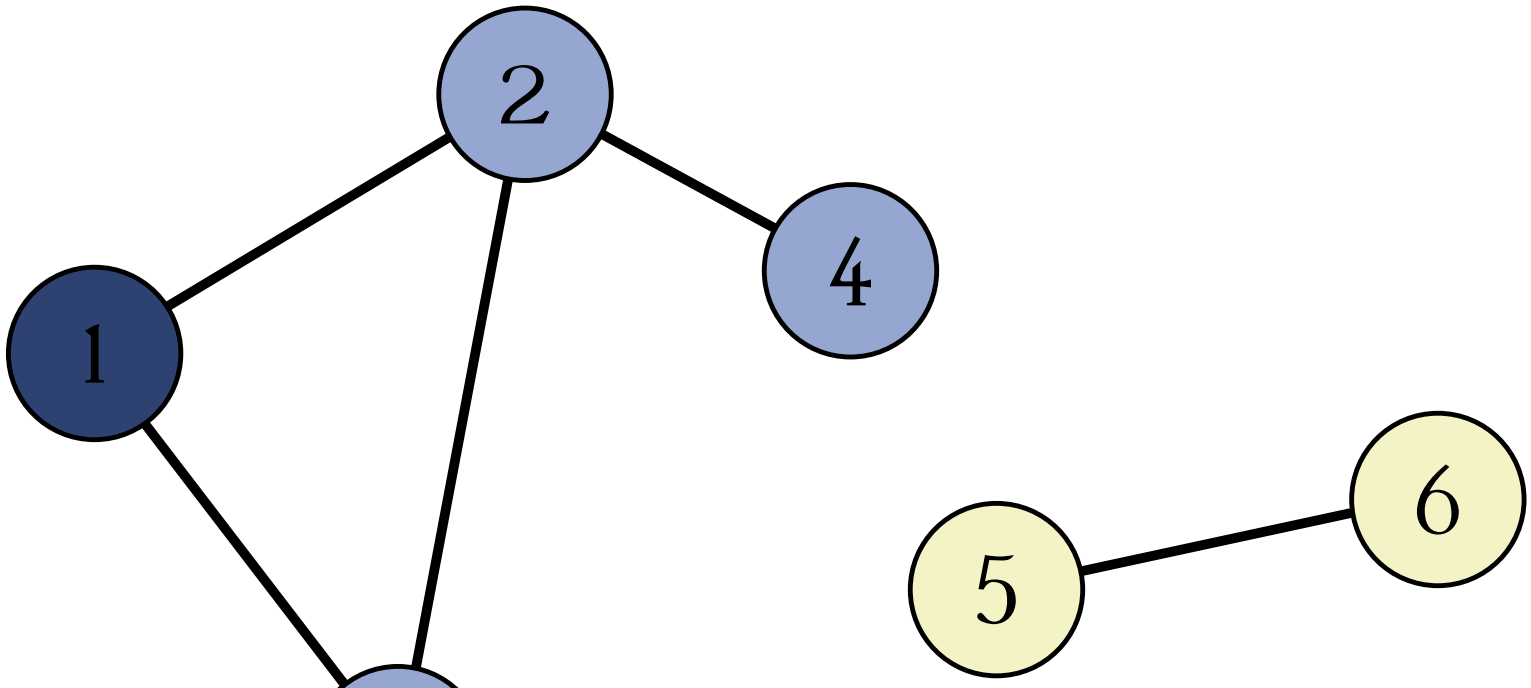
Connected Components Example



Connected Components Example



Connected Components Example



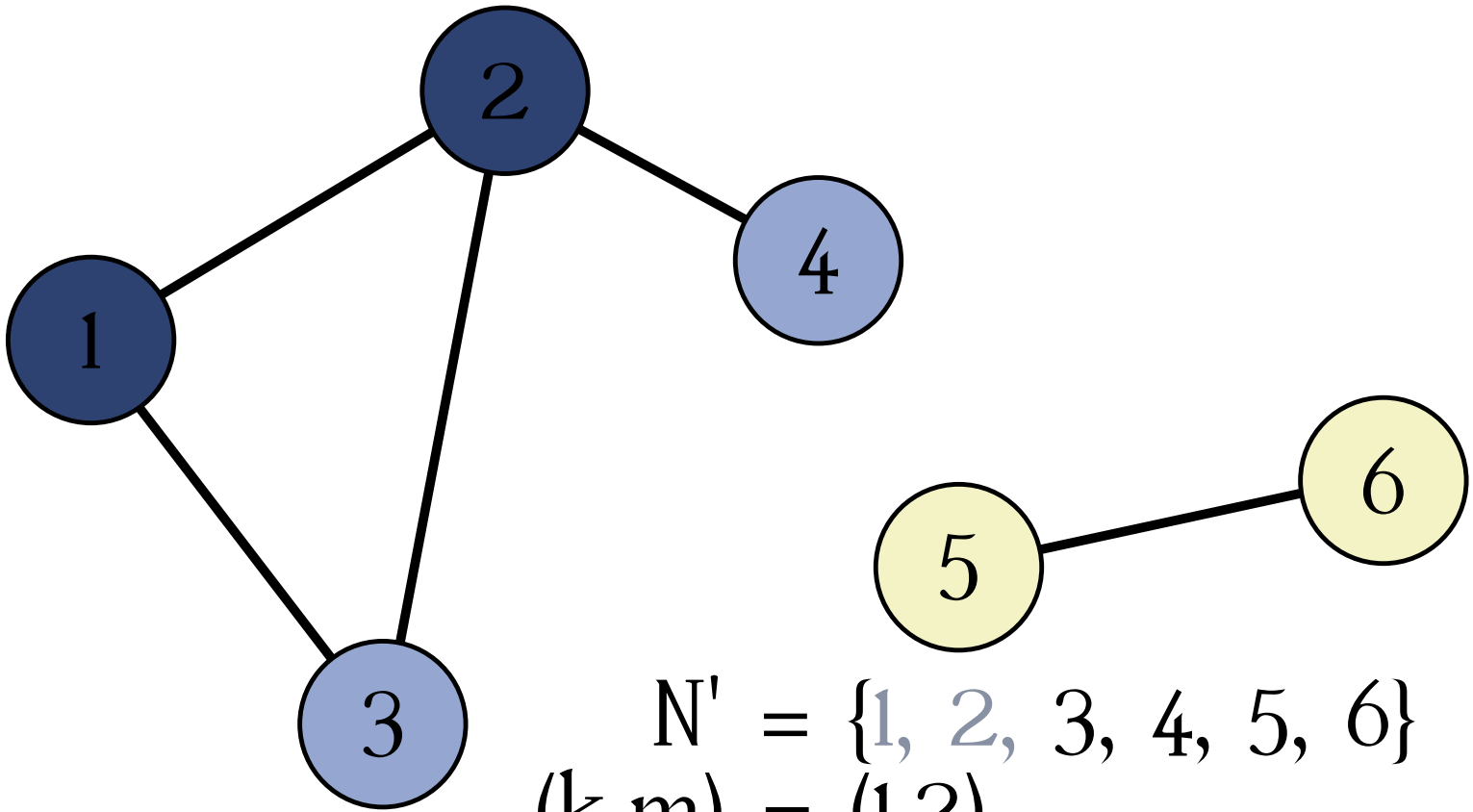
$N' = \{1, 2, 3, 4, 5, 6\}$

$i = 1$

$C_1 = \{1\}$

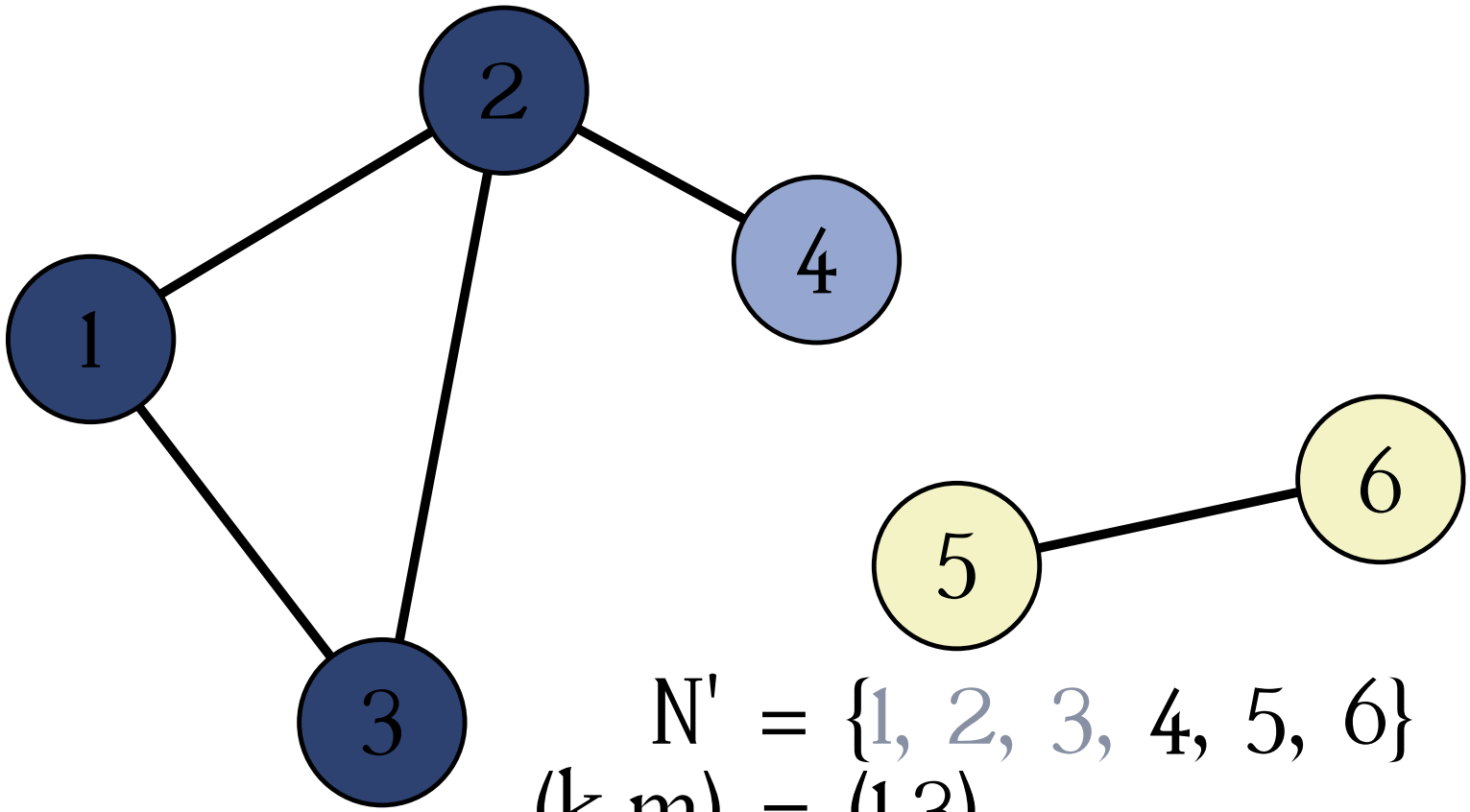
$L = \{(1,2), (1,3)\}$

Connected Components Example



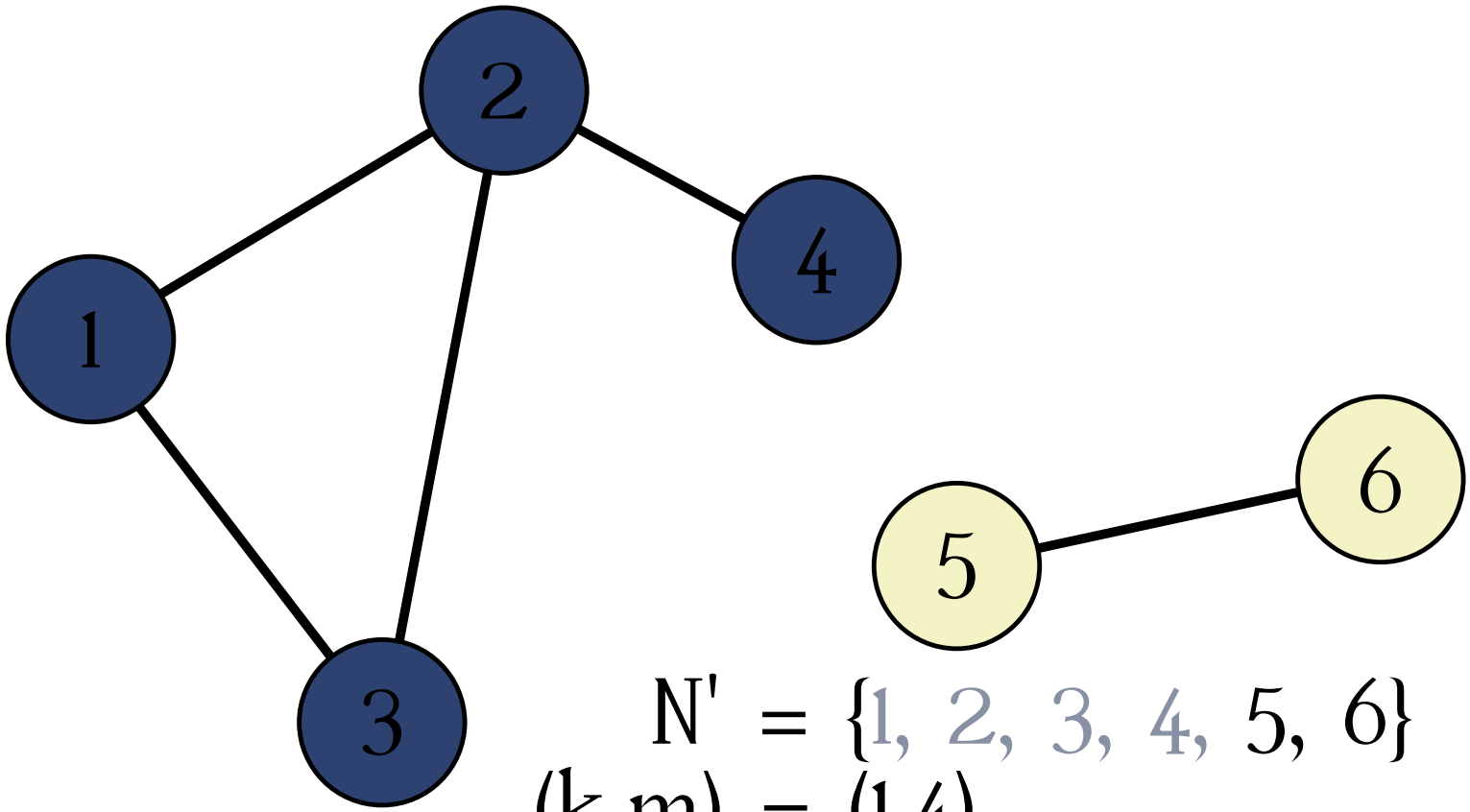
$$N' = \{1, 2, 3, 4, 5, 6\}$$
$$(k,m) = (1,2)$$
$$C_1 = \{1, 2\}$$
$$L = \{(1,2), (1,3), (2,4)\}$$

Connected Components Example



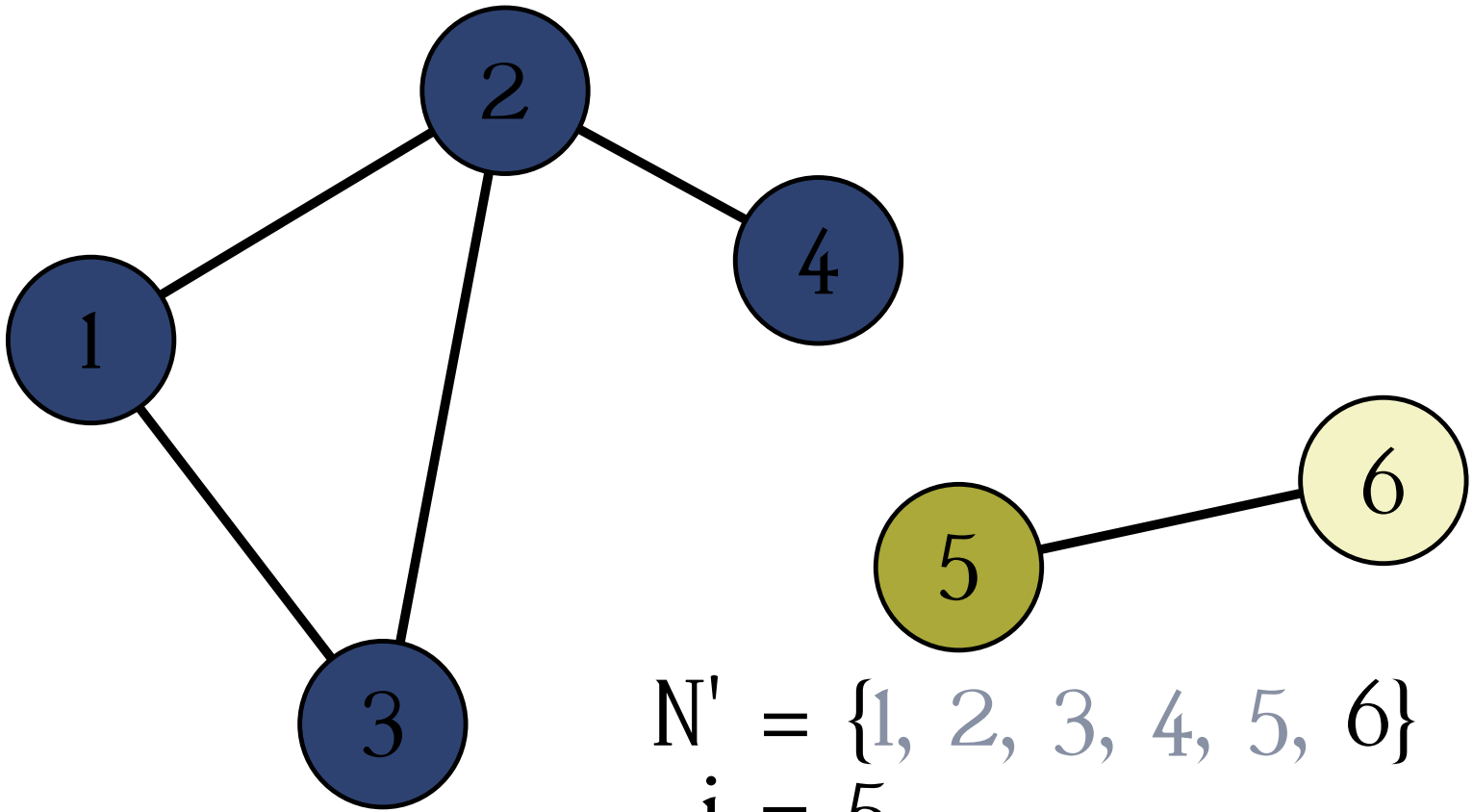
$$N' = \{1, 2, 3, 4, 5, 6\}$$
$$(k,m) = (1,3)$$
$$C_1 = \{1, 2, 3\}$$
$$L = \{(1,2), (1,3), (2,4)\}$$

Connected Components Example

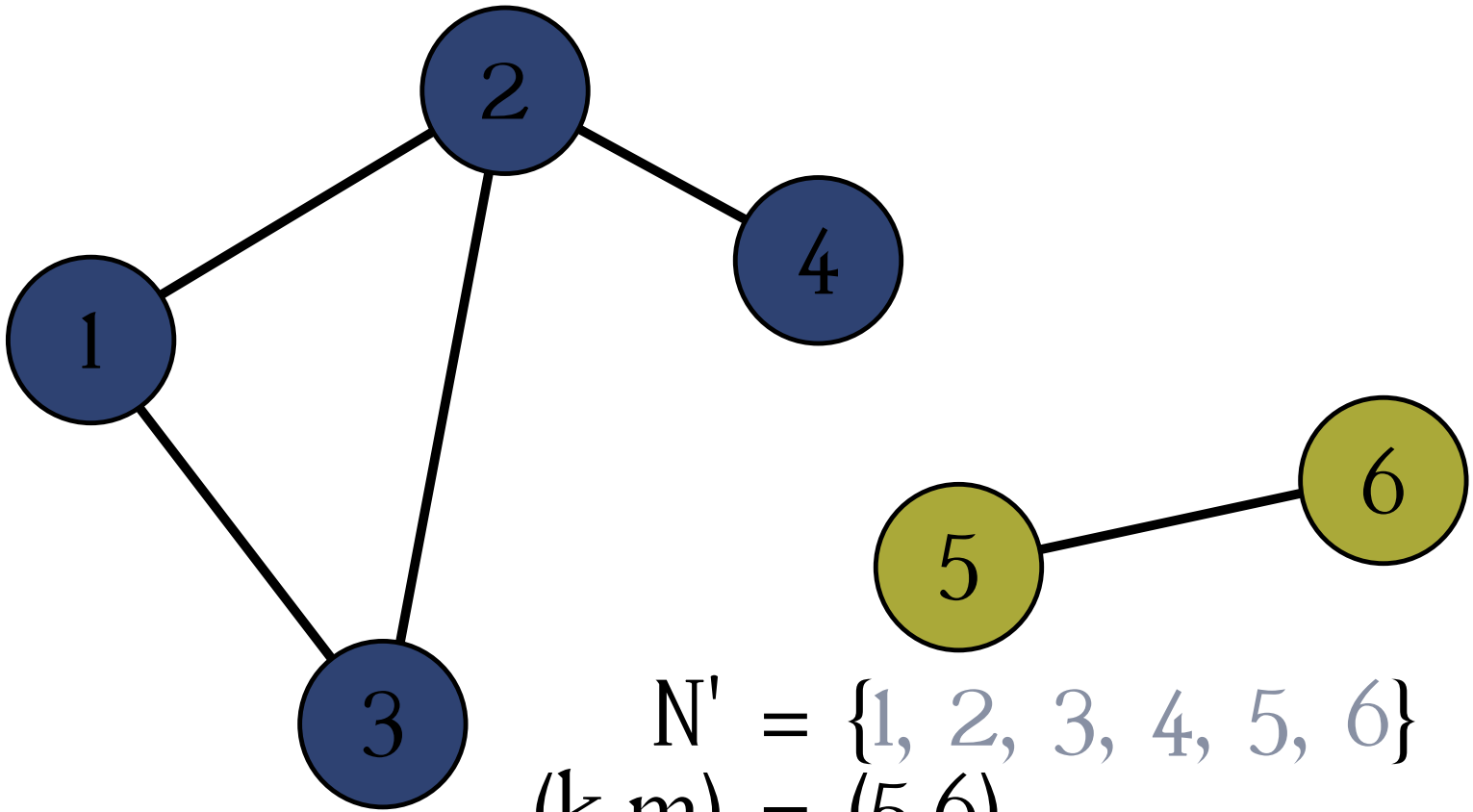


$$N' = \{1, 2, 3, 4, 5, 6\}$$
$$(k,m) = (1,4)$$
$$C_1 = \{1, 2, 3, 4\}$$
$$L = \{(1,2), (1,3), (2,4)\}$$

Connected Components Example



Connected Components Example



$$N' = \{1, 2, 3, 4, 5, 6\}$$
$$(k,m) = (5,6)$$
$$C_5 = \{5, 6\}$$
$$L = \{ (5,6) \}$$

Application 1

Key requirements for critical infrastructure networks (*e.g.*, Internet, Water, Power, ...)

- Reliability

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Application 1

The simplest definition of "reliability" used in networks is some variant of k-connectedness

- The particular variant depends on the failure modes of the network
  - do the nodes fail, or the edges (or both)?
- Leads to network designs with redundancy
  - not necessarily k-fold redundancy
- Is this a good enough definition of reliability

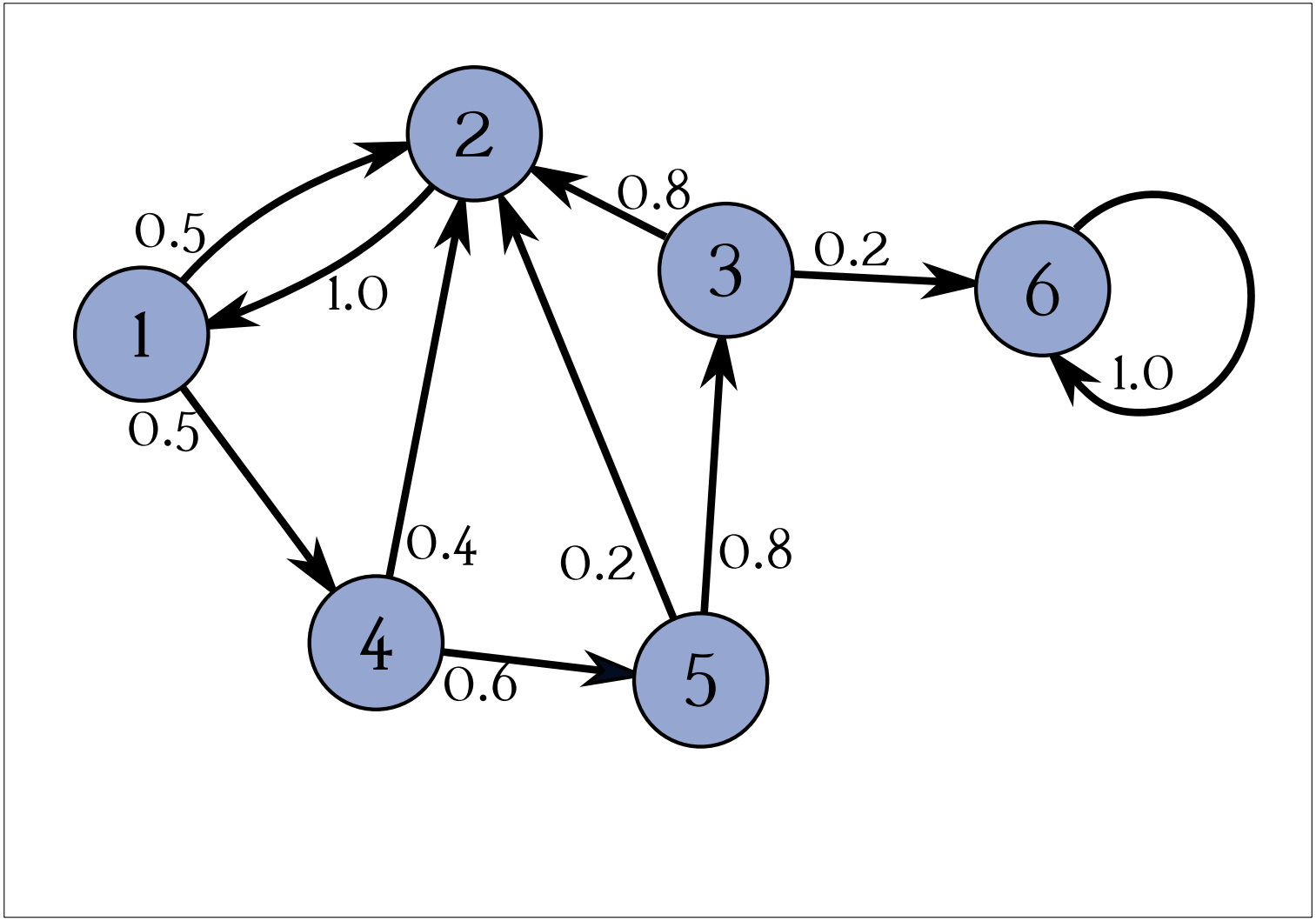
Application 2

Markov chains probability transition matrix

$$P = \begin{pmatrix} 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.8 & 0.4 & 0.2 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.8 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.6 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.2 & 0.0 & 0.0 & 1.0 \end{pmatrix}$$

Application 2

Markov chains are described by a directed graph with self-loops, e.g.,



The transition matrix is just a weighted adjacency matrix.

Application 2

First step of studying a Markov chain is to check its properties

Definition

State i is *accessible* from state j if it is possible to get from i to j.

Accessible = a path from i to j exists

Definition

A Markov chain is *irreducible* if it is possible to get to any state from any state.

Irreducibility = strong connectivity of the graph

Definition

A *communicating class* is a maximal set of mutually accessible states.

Communicating class = connected component

Section 2

Graph Traversal

- Connected components as described above has a little vagueness
  - when I say "choose" how do you choose?
- It's an example of *graph traversal*
  - where we want to visit each node of a graph (at least once)
  - ordered nodes by connectivity
- Traversals used for lots of algorithms
  - could be to search for an element
  - or to calculate a value for each node

  Maybe you don't have the whole graph stored in memory, but have to read bits, e.g., traversing Facebook graph
- There are two main strategies
  - Depth-First Search
  - Breadth-First Search
- For the sake of simplicity, we will assume graphs are connected
- Easiest to understand in neighbour-list representation

Depth-First Search

Visit a neighbour's children before you visit the next neighbour

```
1 Function DFS(G, i);
  Input: A Graph G = (N, E), and start node i ∈ N
2 label i as explored;
3 forall j ∈ neighbourhood{i} do
4 |   if j is unexplored then
5 |   |   DFS(G, j);
6 |   end
7 end
```

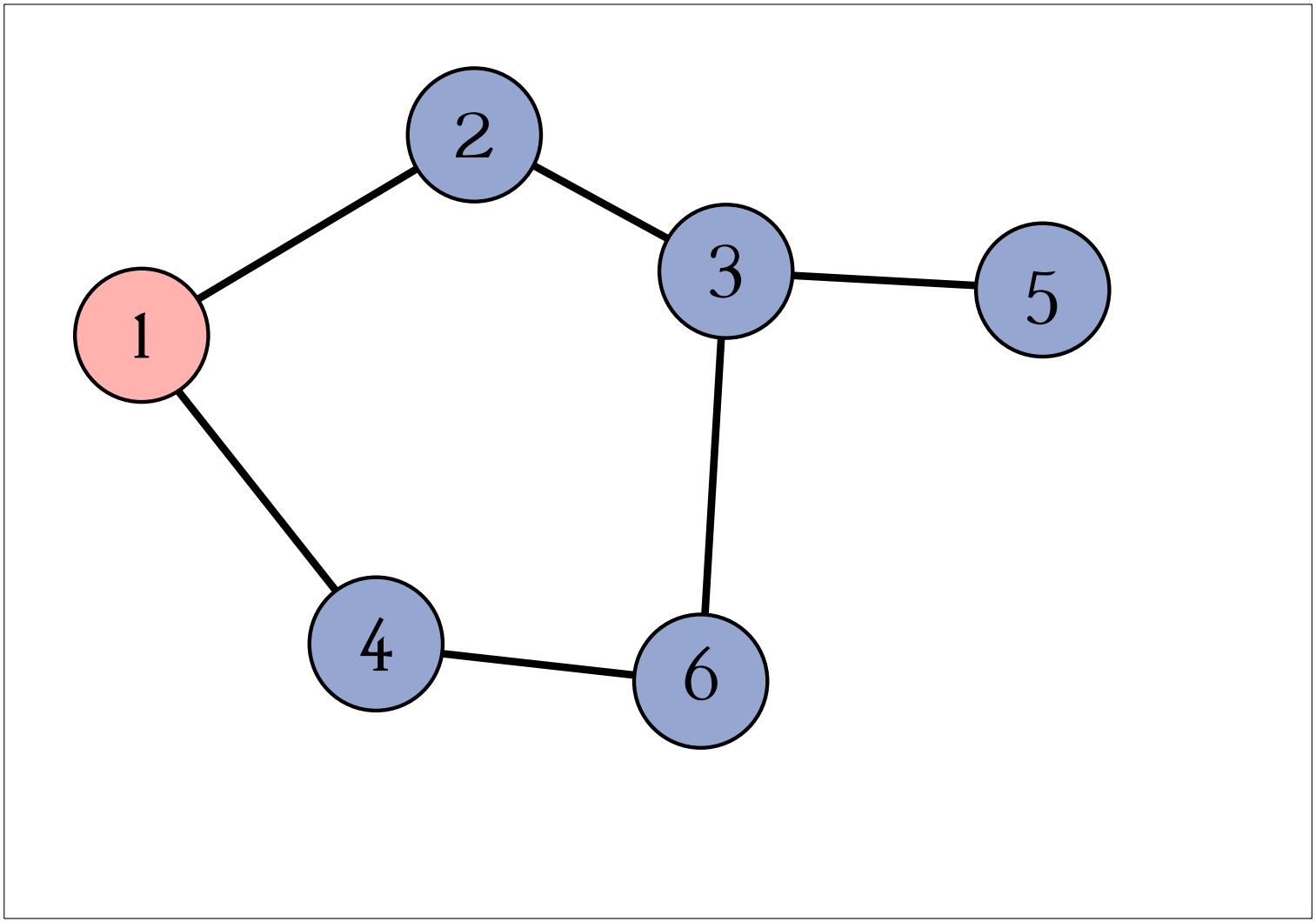
- We could make this faster by avoiding edges going backwards.
- At the moment the algorithm doesn't do anything
  - a *search* also checks something about the node, and returns the first one that checks out
  - but we might also do some sort of update
  - or use to find a connected component ...

Breadth-First Search

Visit all neighbours before you visit their children

```
1  Function BFS(G, i);
   Input: A Graph G = (N, E), and start node i ∈ N
2  label i as explored;
3  create queue Q;
4  put i on Q;
5  while Q not empty do
6  |   take j off the front of Q;
7  |   forall k ∈ neighbourhood{j} do
8  |   |   if k is unexplored then
9  |   |   |   label k as explored;
10 |   |   |   put k on Q;
11 |   |   end
12 |   end
13 end
```

Depth-First Search Example



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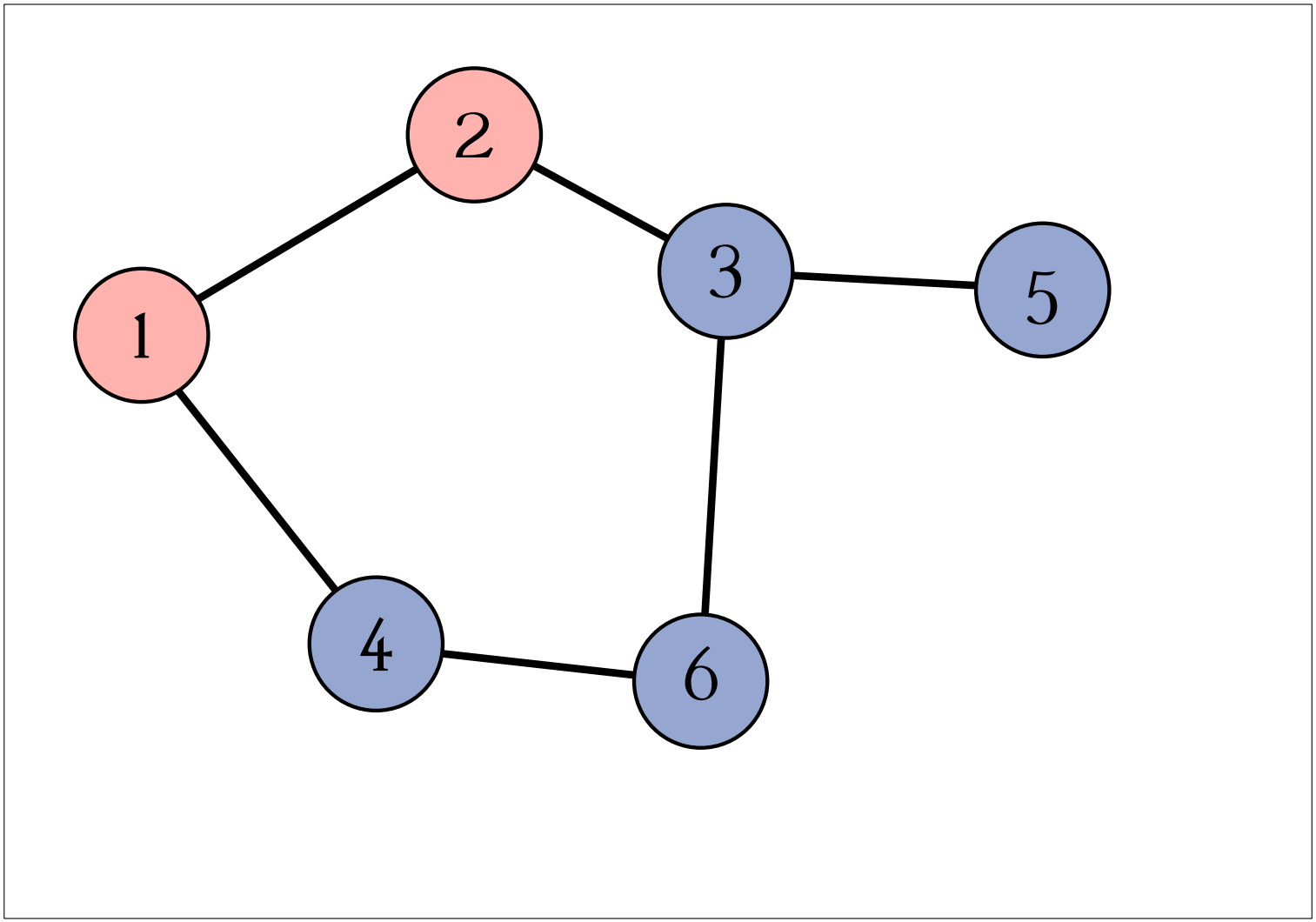

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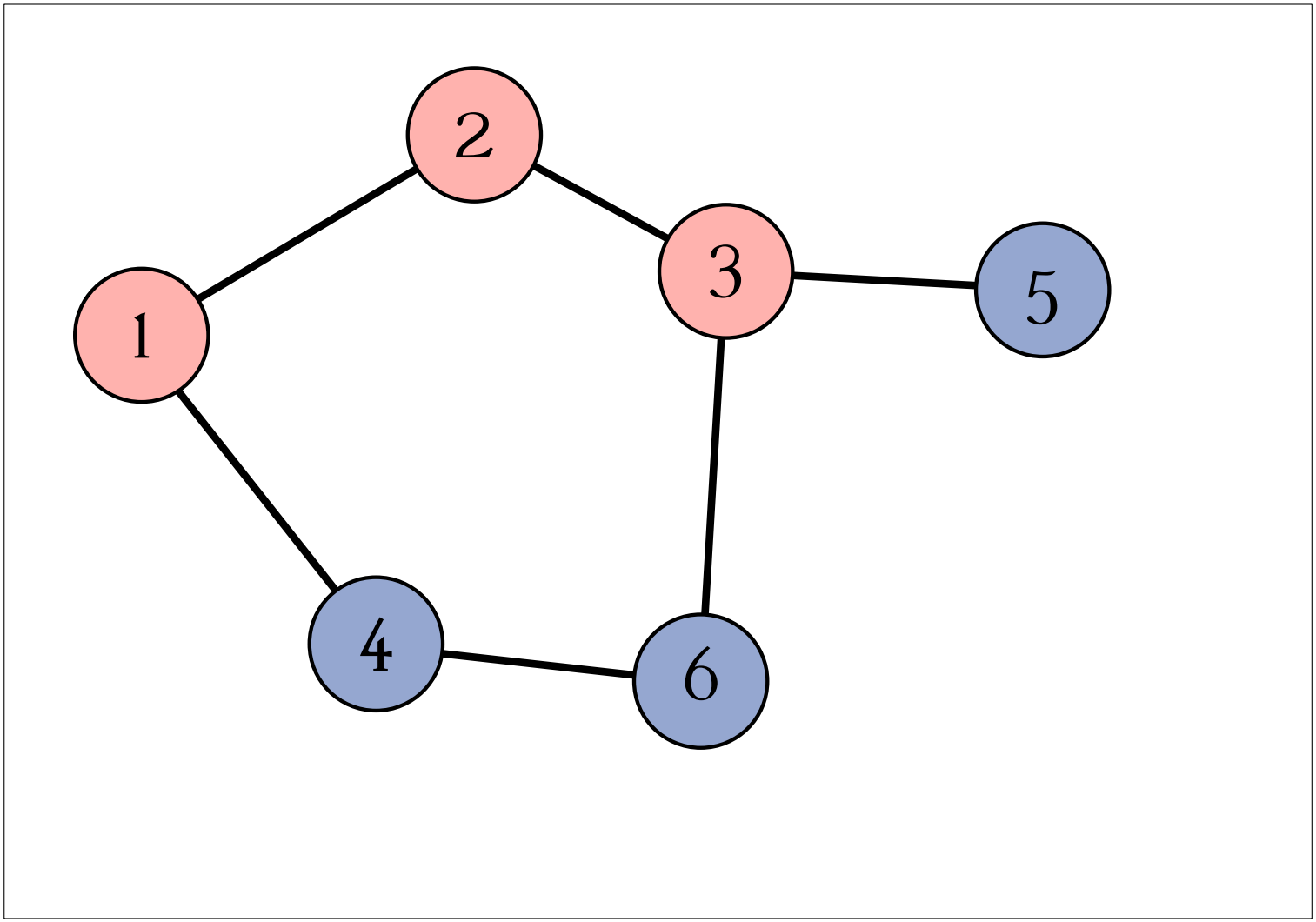

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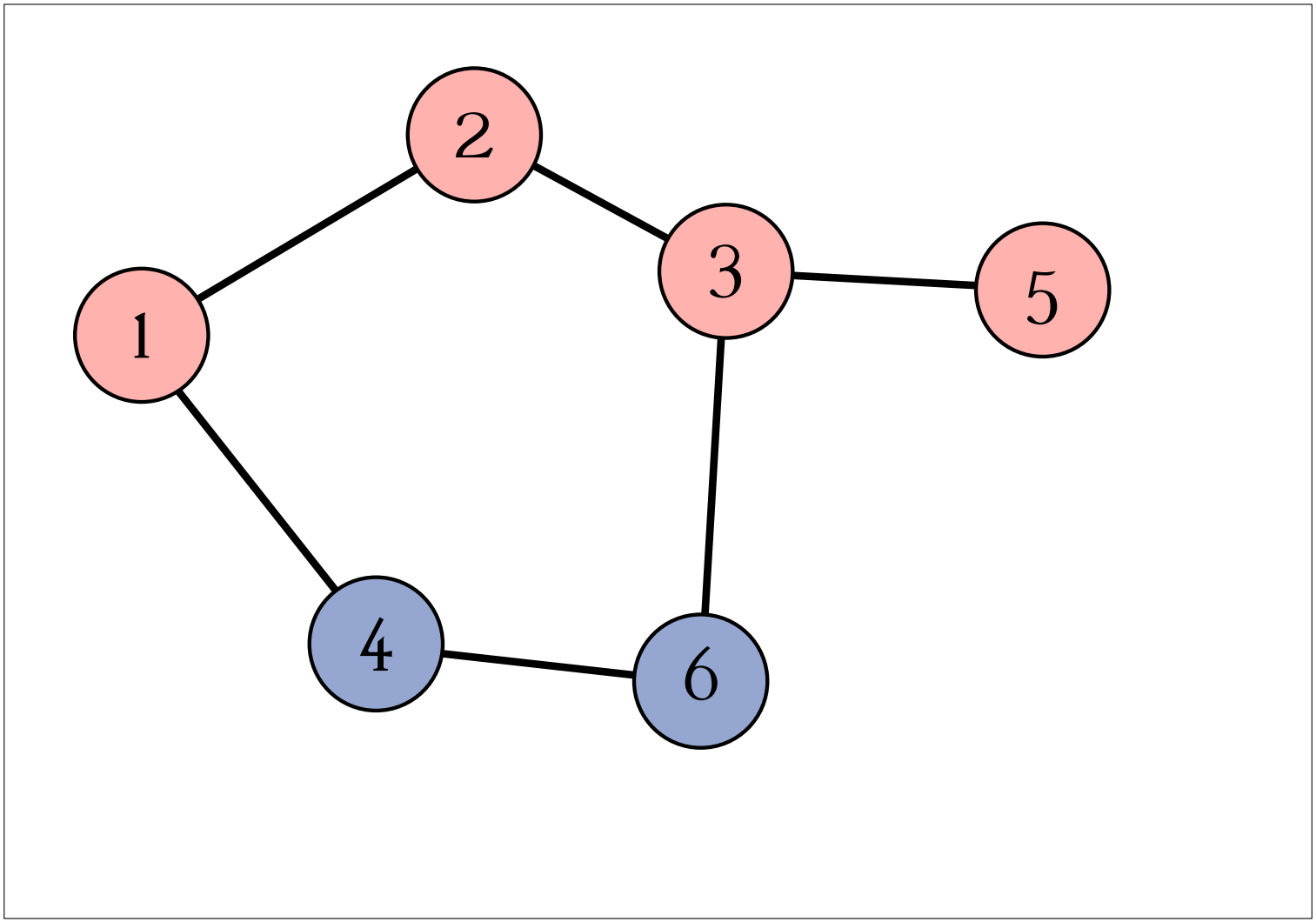

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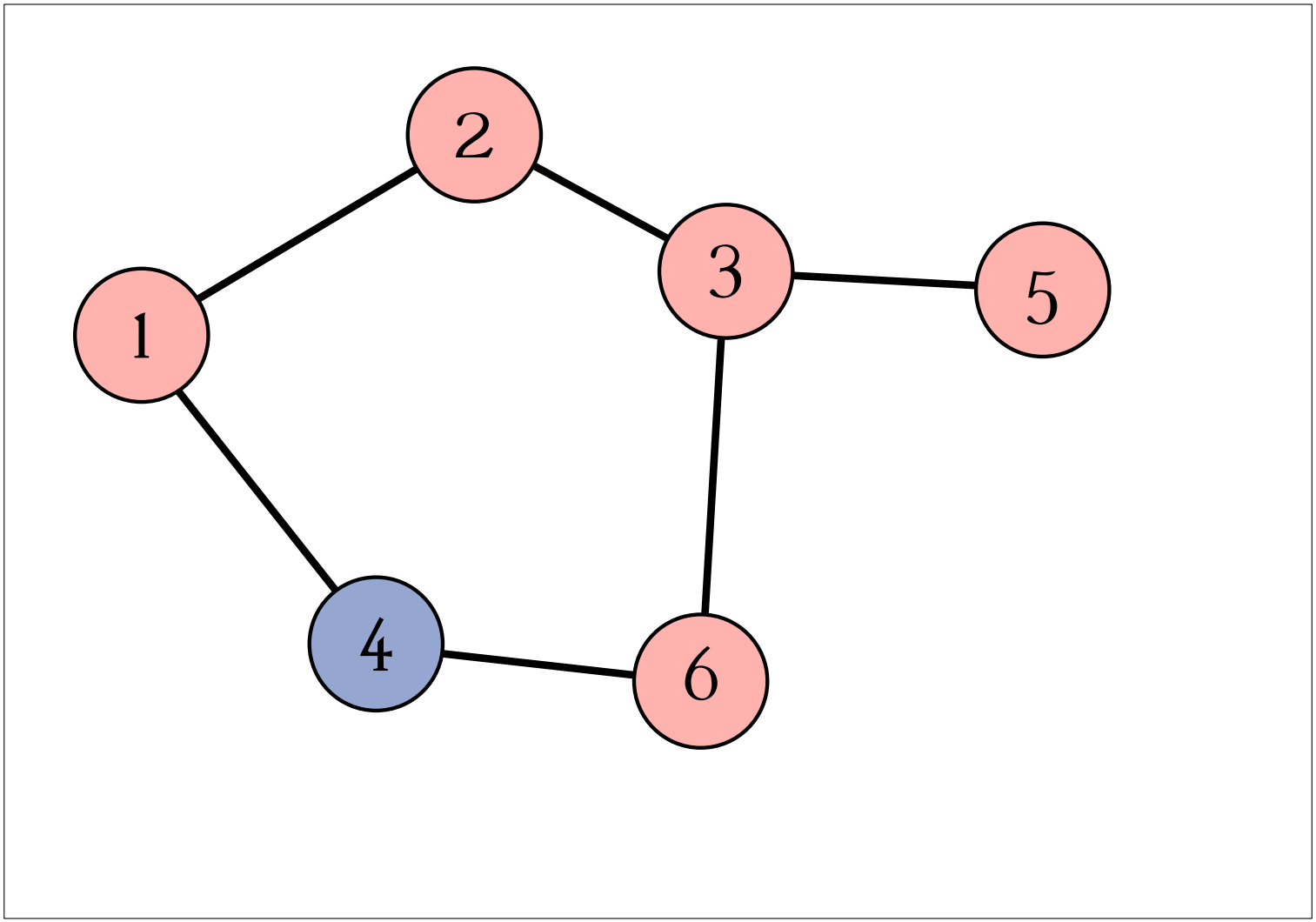

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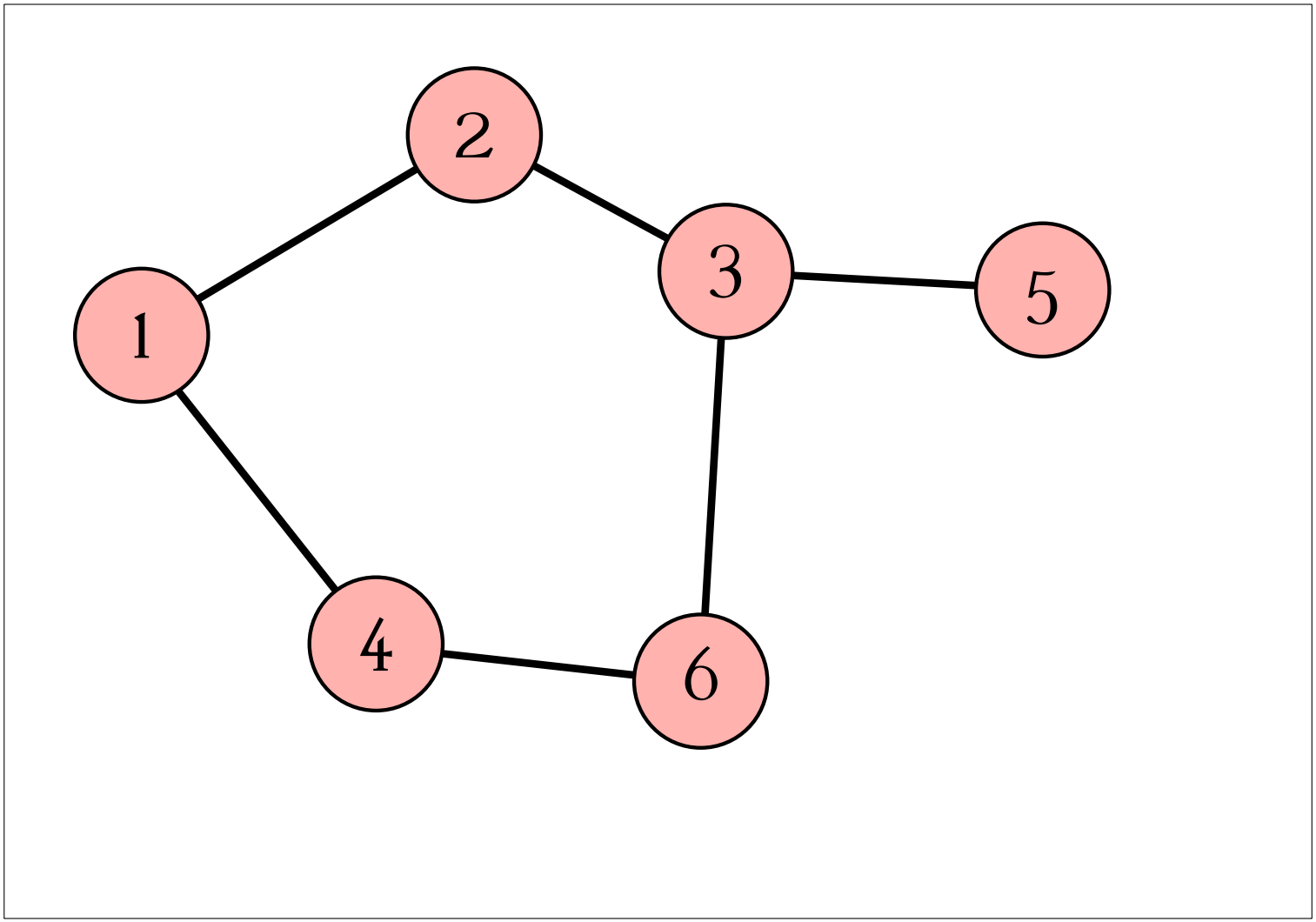

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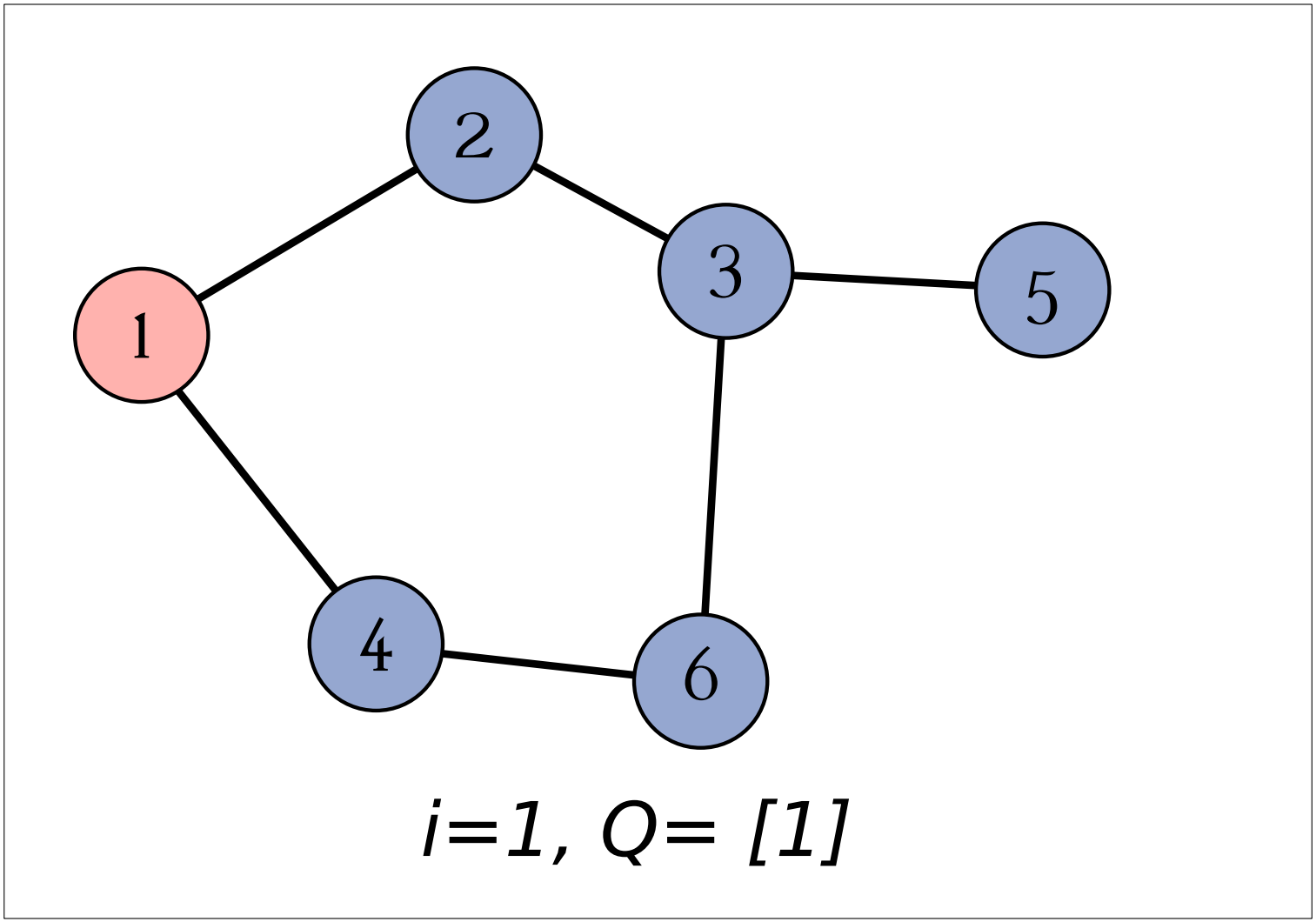

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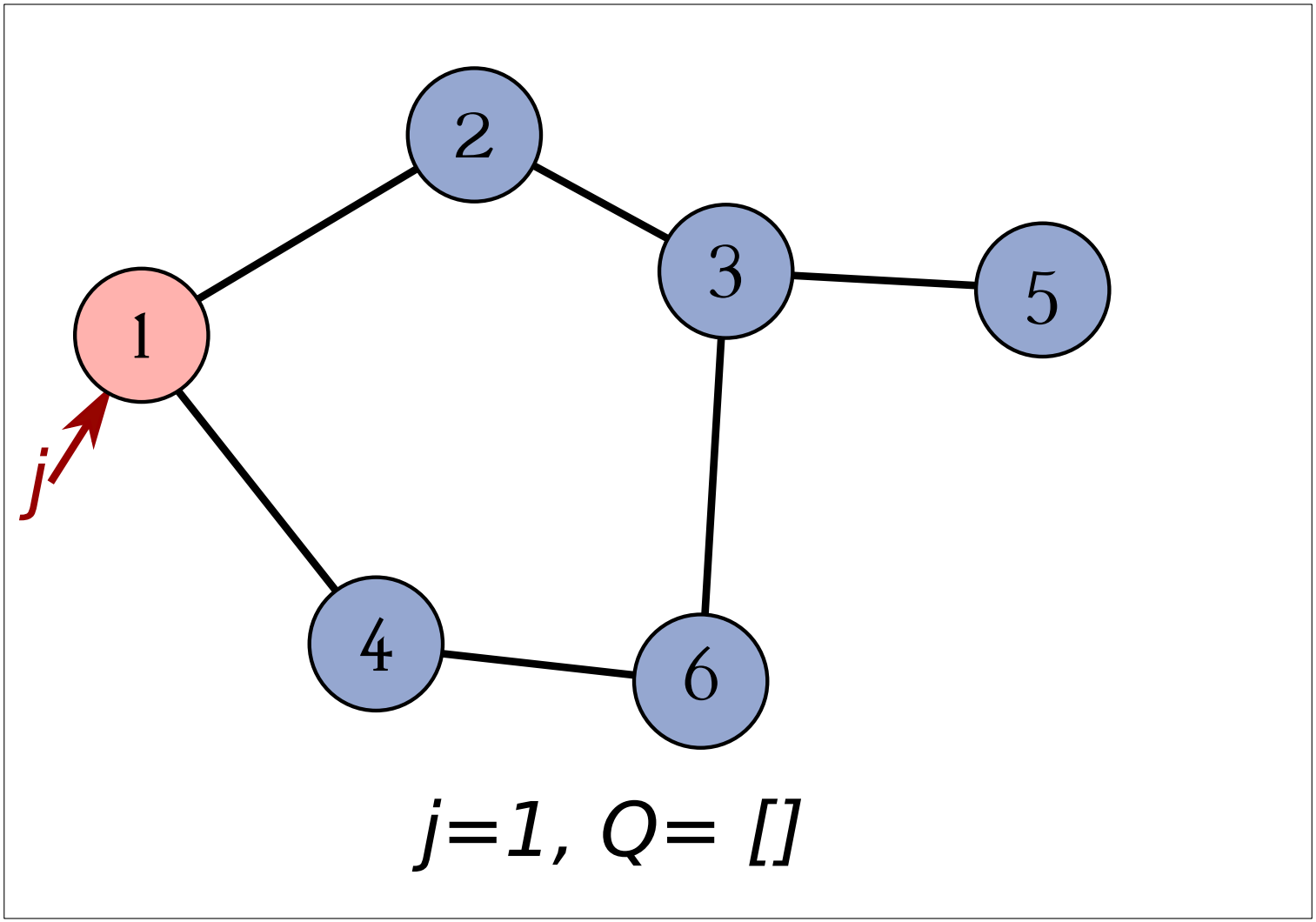

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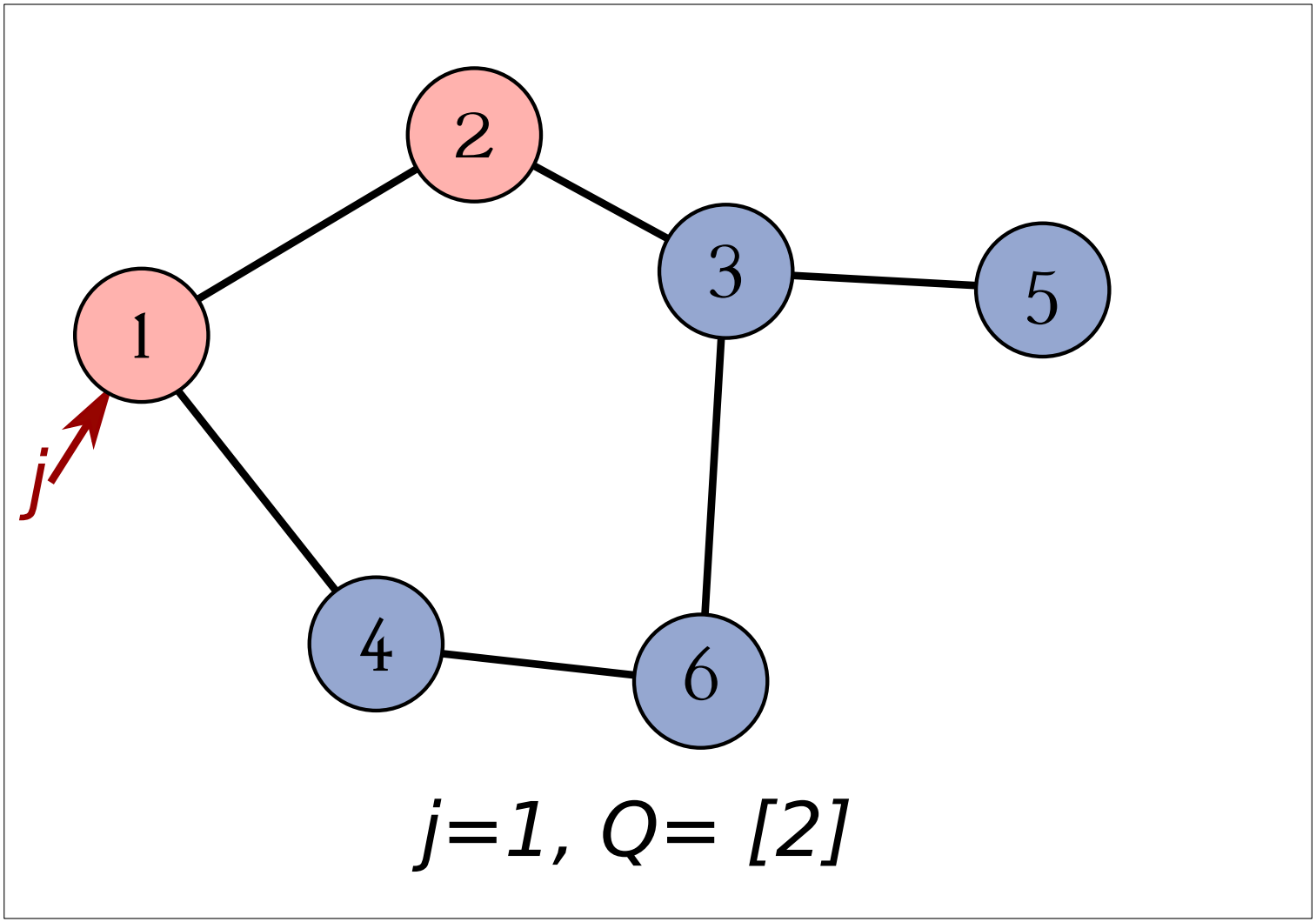

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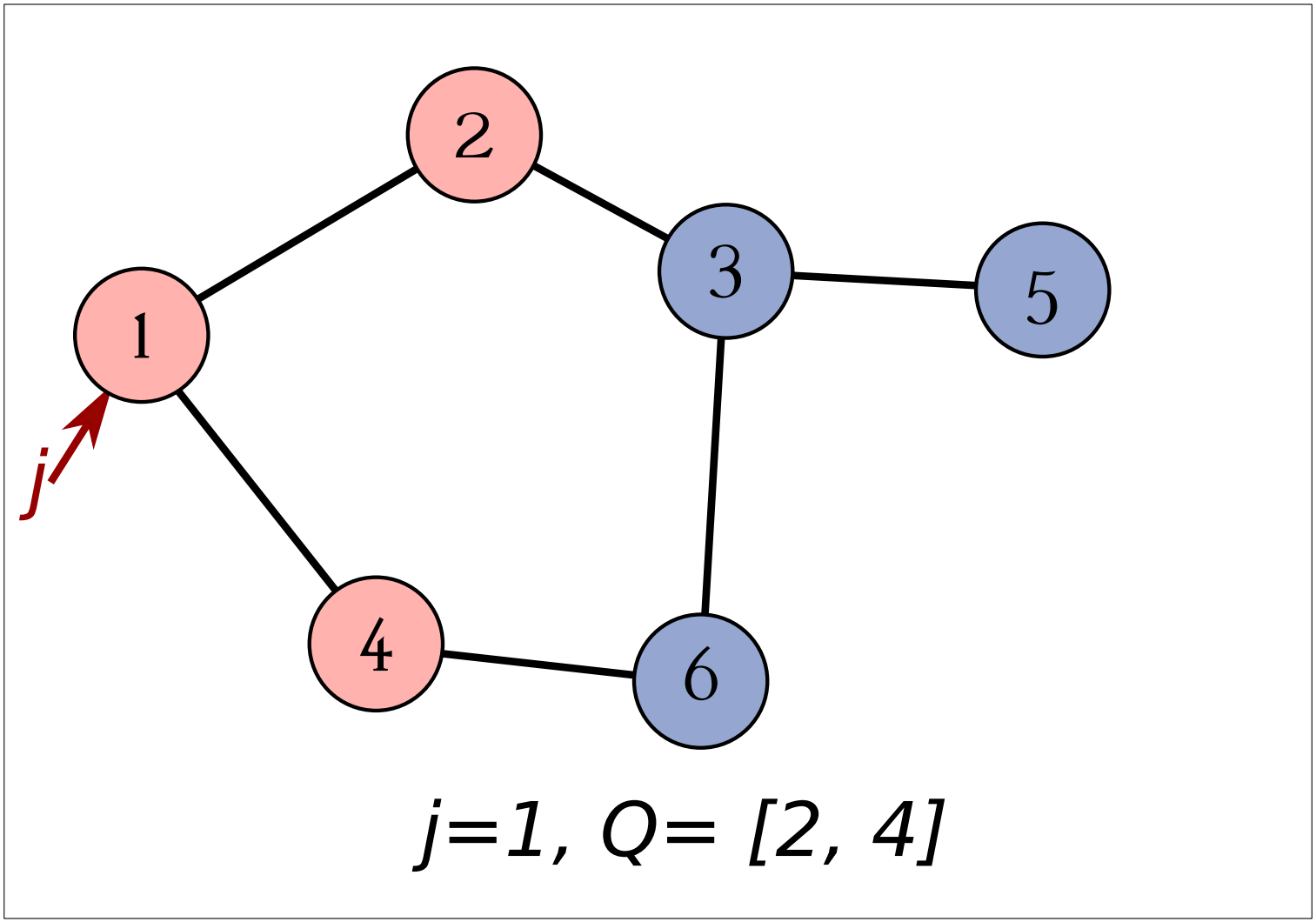

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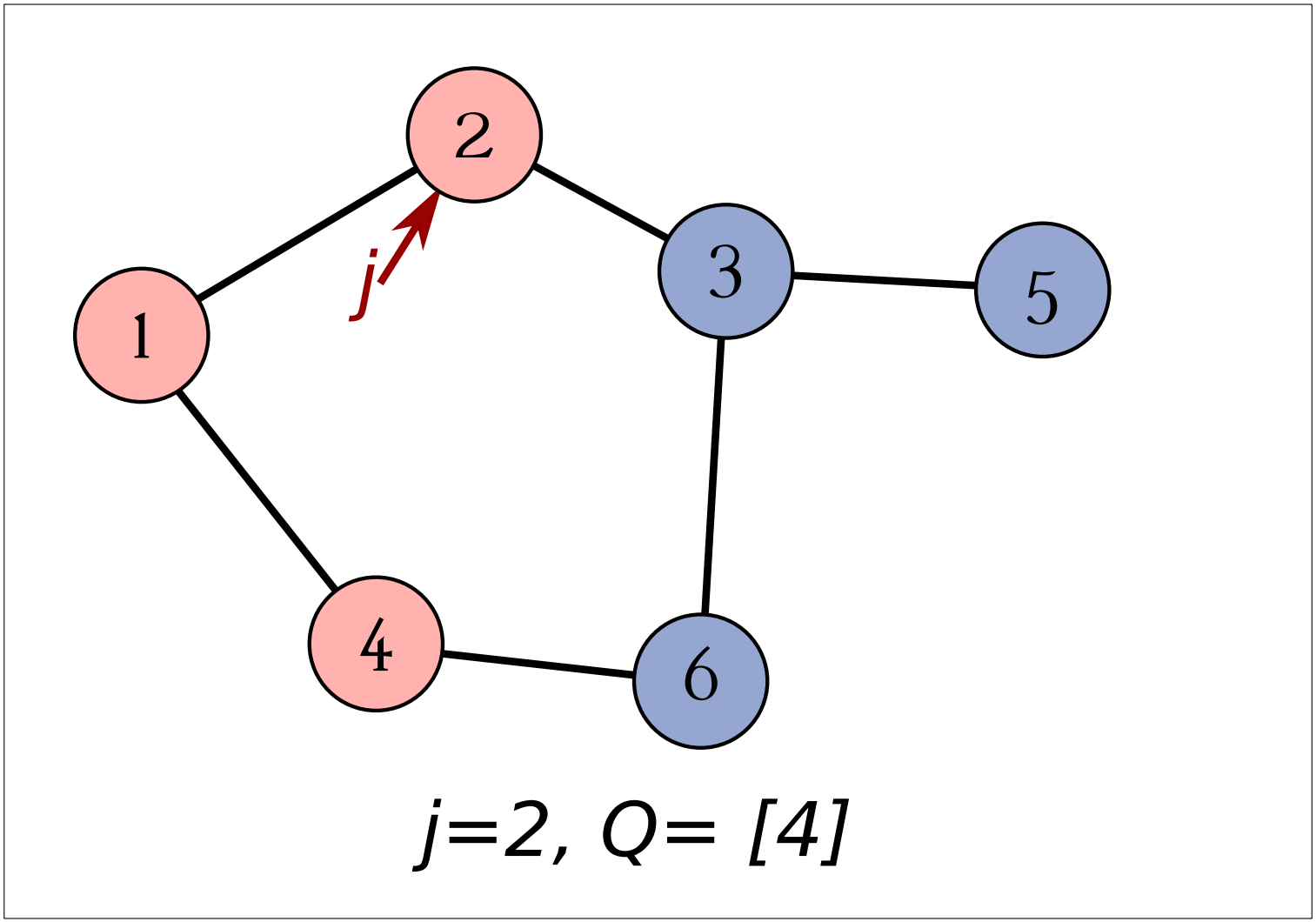

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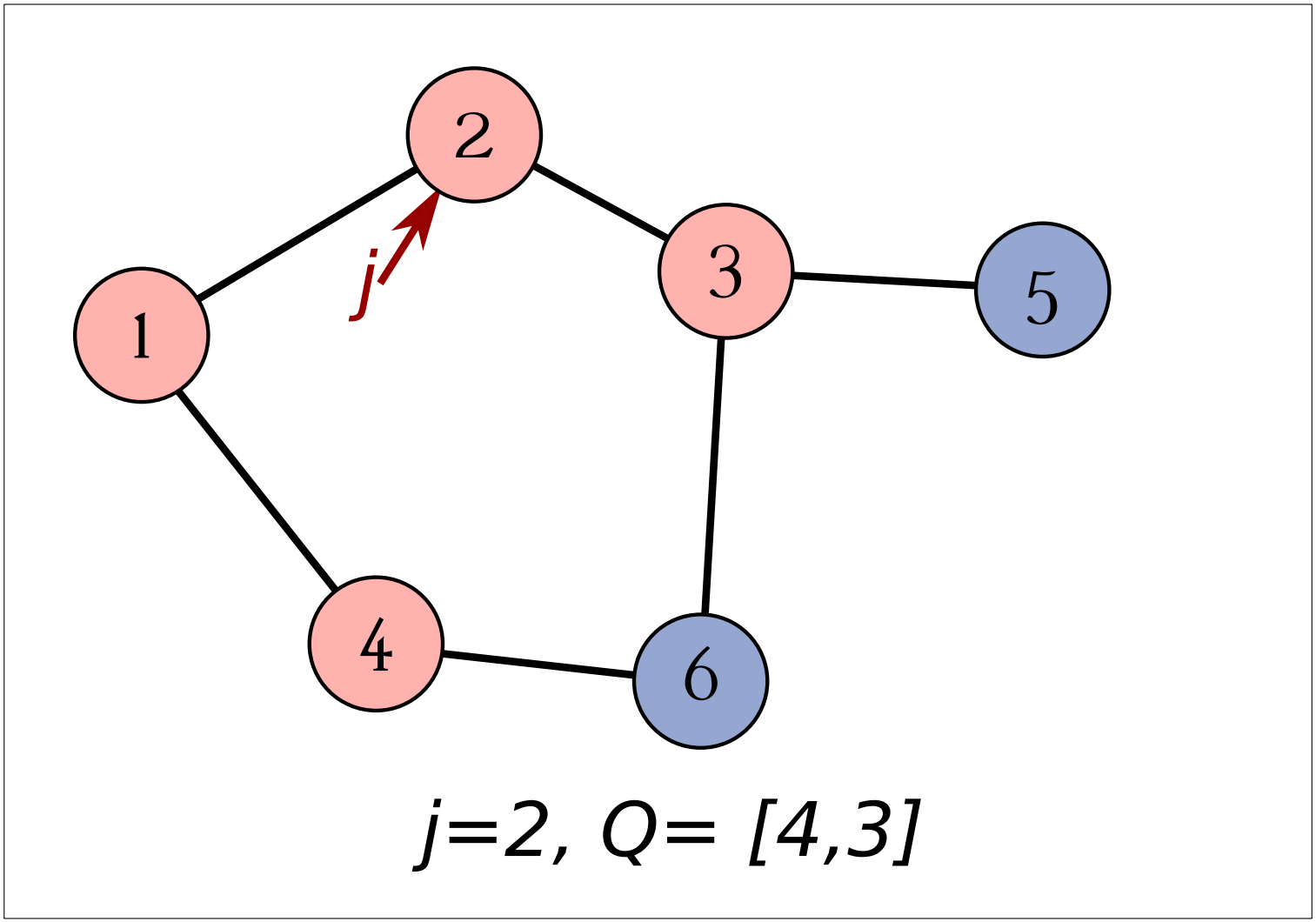

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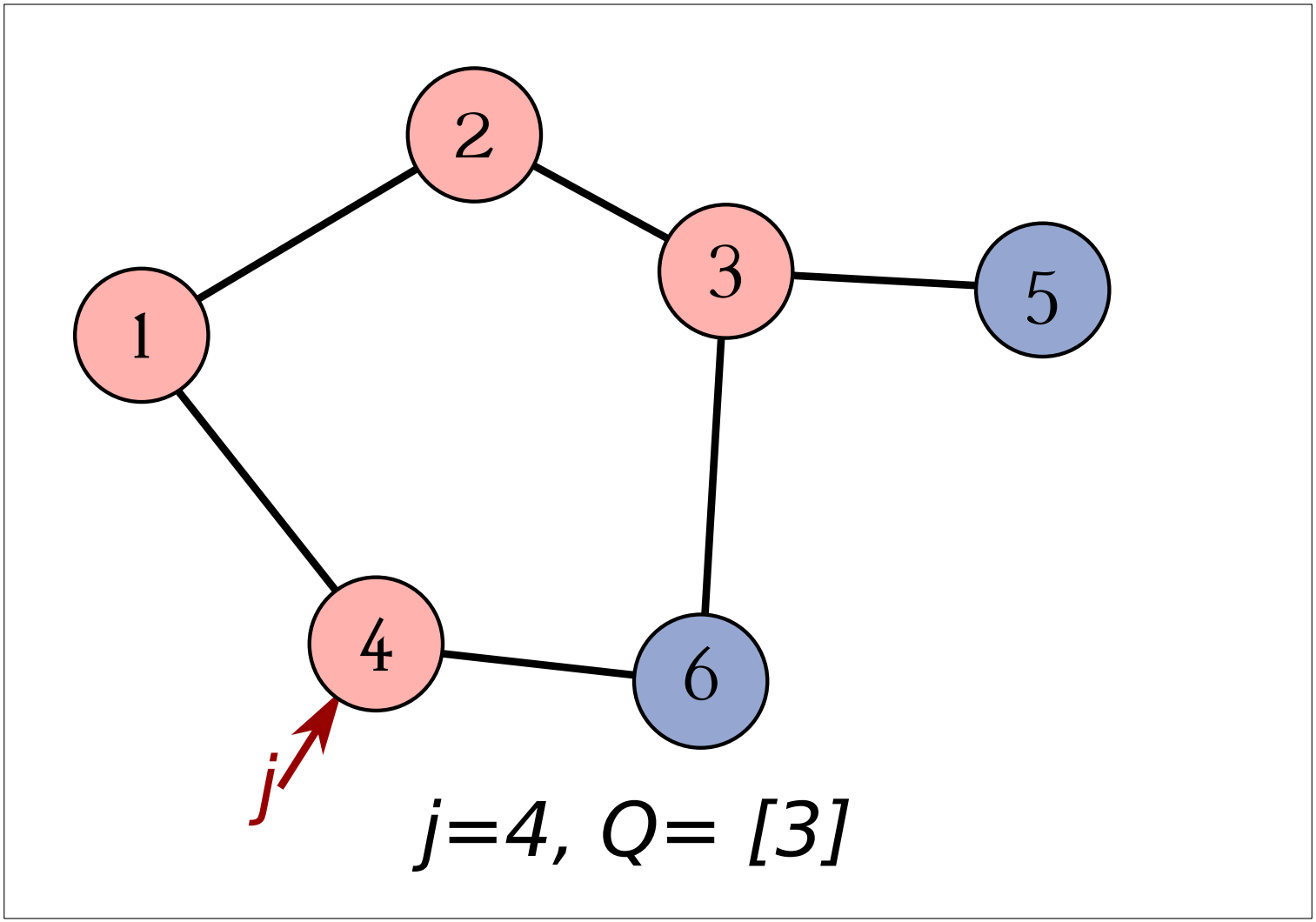

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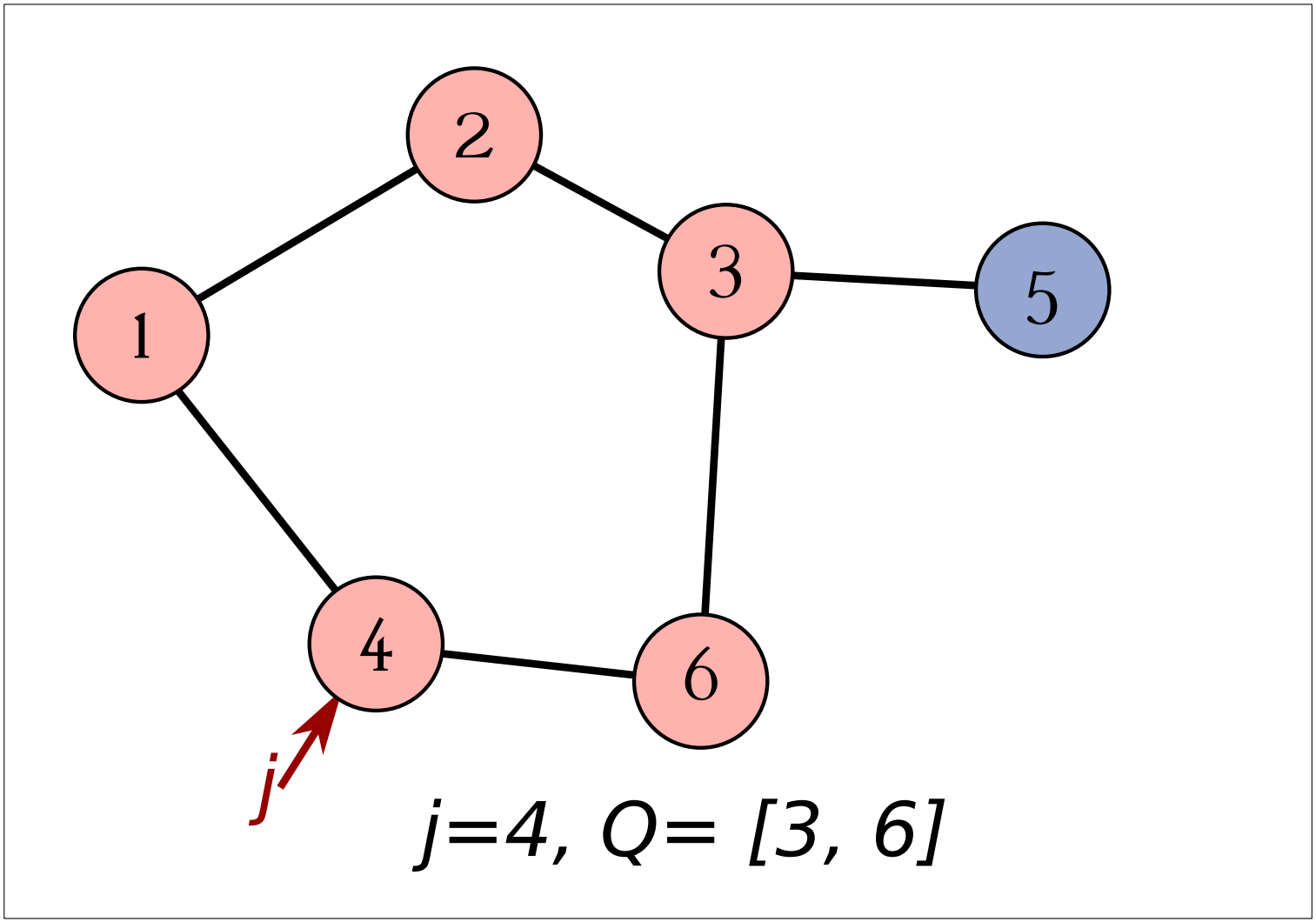

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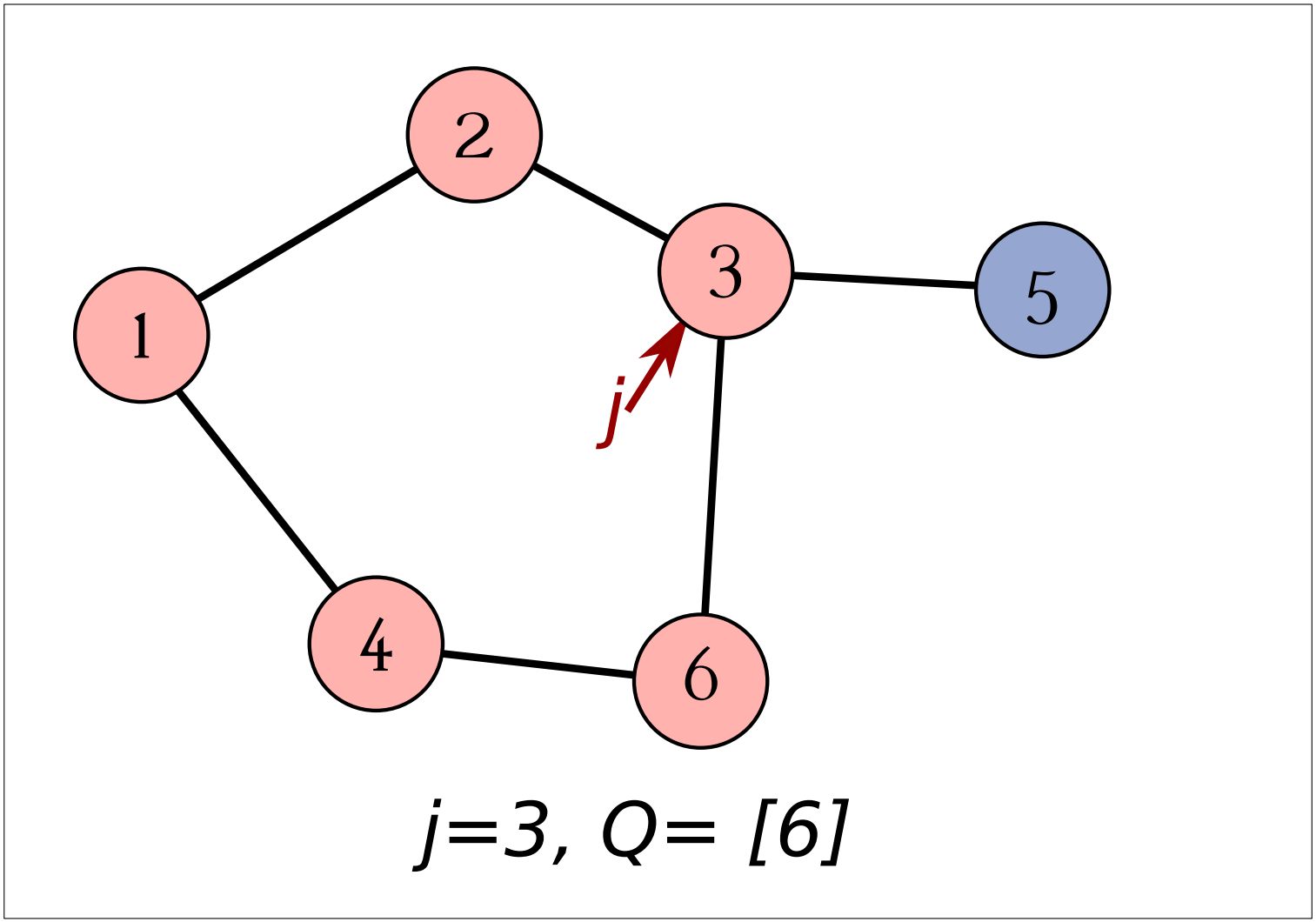

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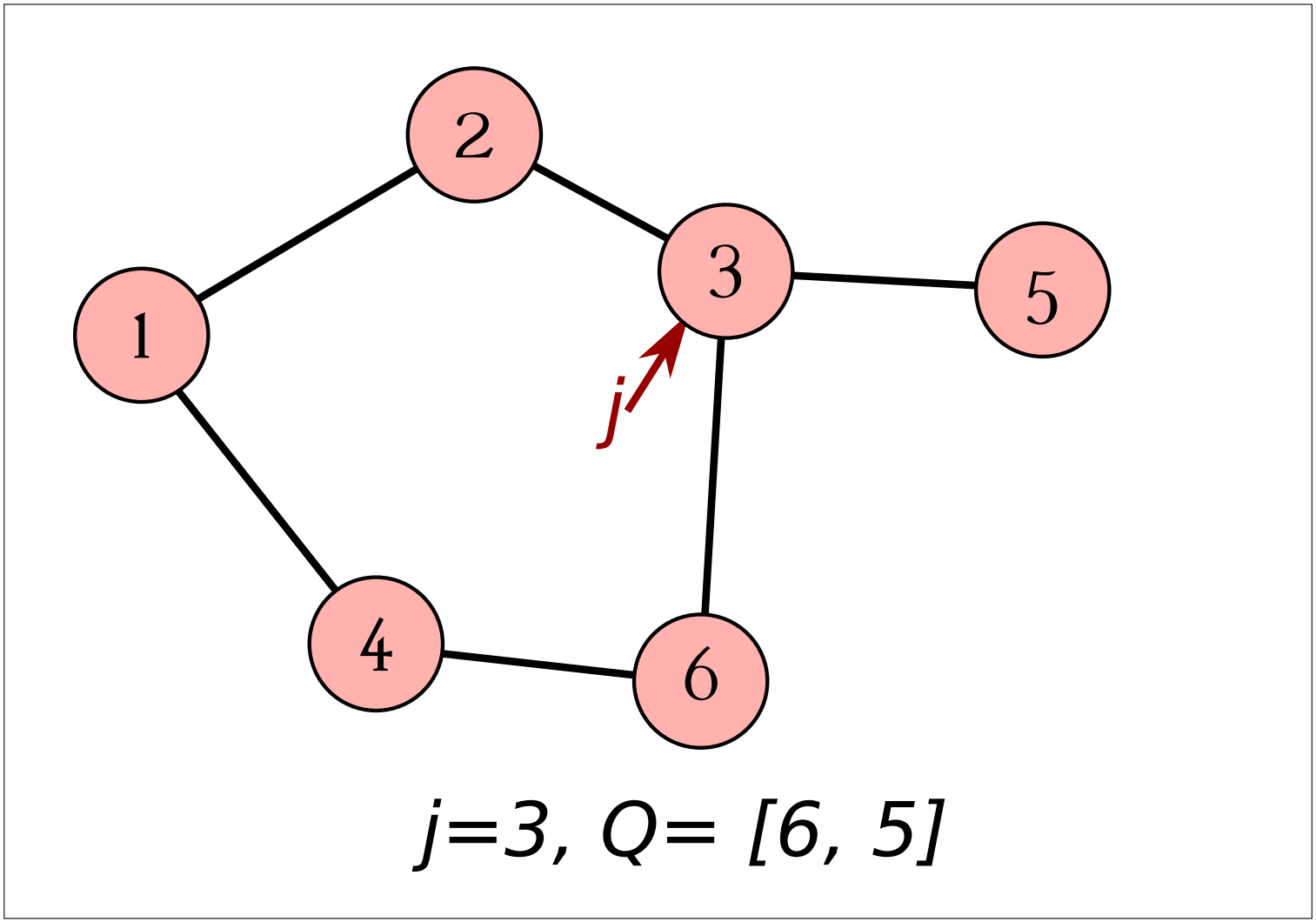

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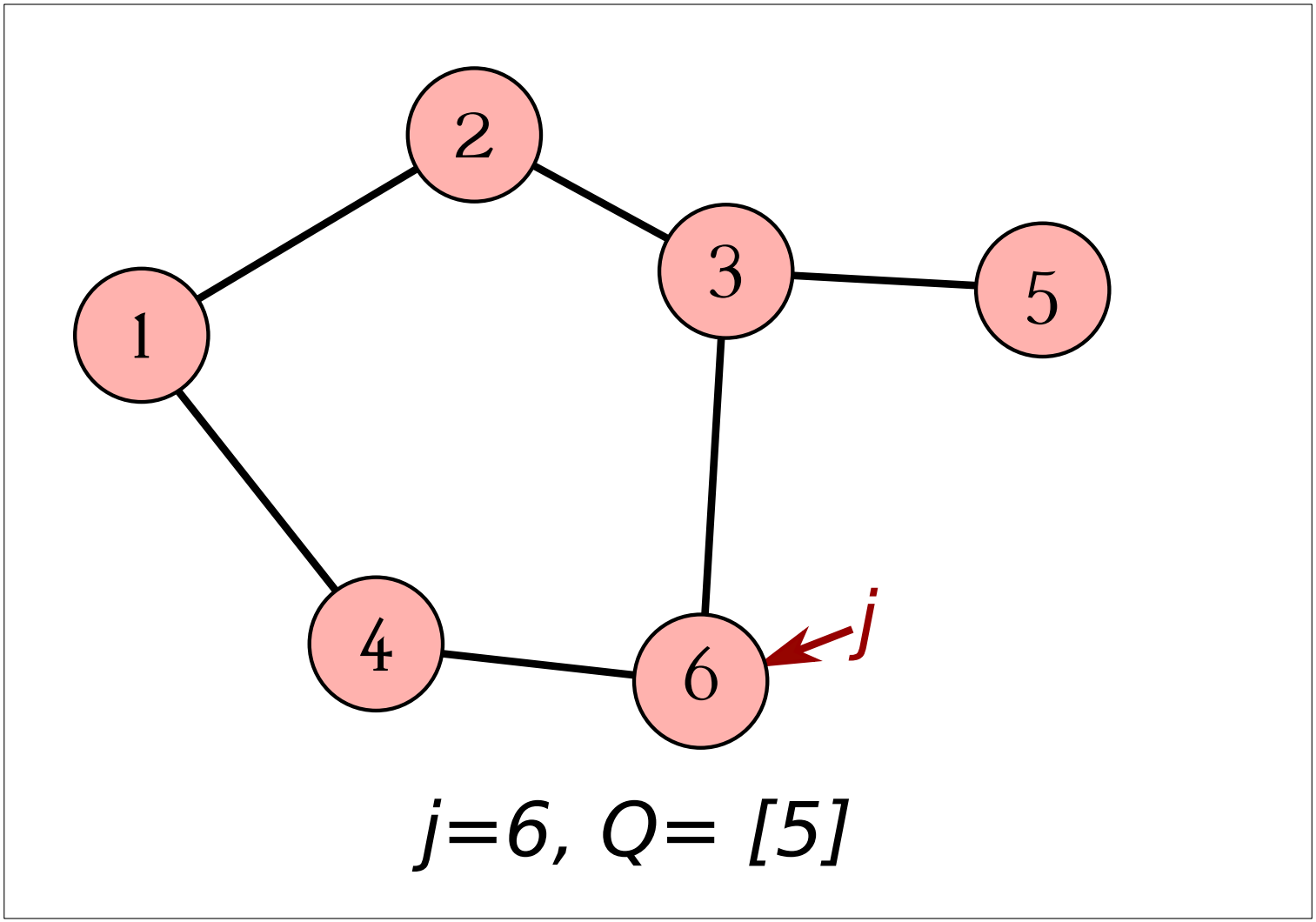

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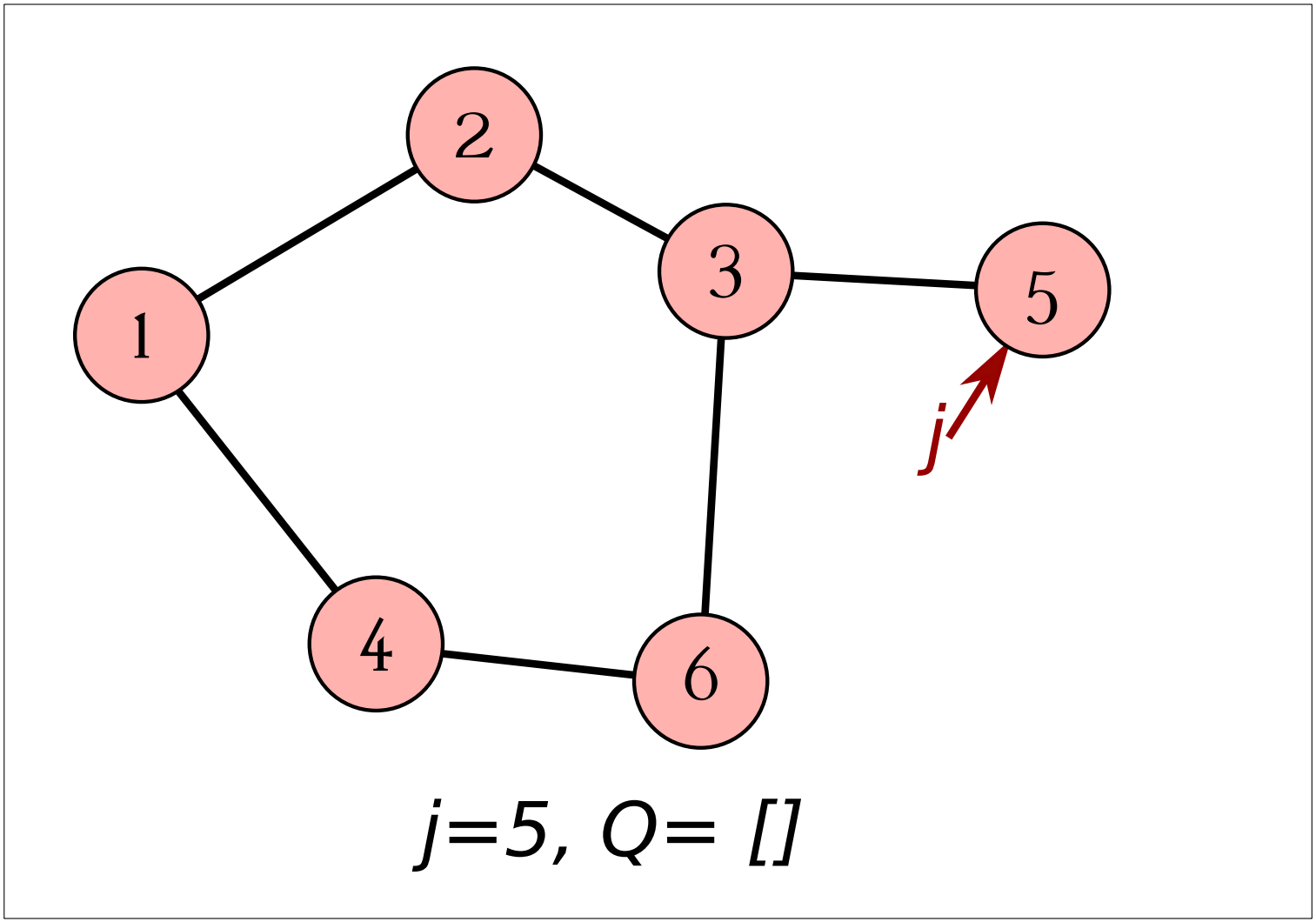

Further reading I