Complex-Network Modelling and Inference

Lecture 4: Graph connectivity and traversal

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Section 1

Connectivity

Connectivity

Definition

Two nodes are *connected* if a path exists between them.

Definition

A graph is *connected* if all pairs of nodes are connected.

Definition

A *strongly connected* digraph is connected in the sense above, whereas a *weakly connected* digraph is connected if we include all reverse links.

Definition

A graph is *k-edge connected* if the graph remains connected after the removal of any set of $k-1$ edges, and *k-node connected* if the graph remains connected after the removal of any set of $k-1$ nodes.

Definition (Cut)

A *cut* is a partition of the nodes of a graph into two subsets $C = (S, T)$.

Definition (Cut-set)

The *cut-set* of a cut $C = (S, T)$ is the set of edges

$$\{(u, v) \in E | u \in S, v \in T\}$$

i.e., the edges that cross the cut.

Definition (Edge Cut)

An (minimum) *edge cut* is the minimum number of edges whose removal disconnects two nodes i and j, *i.e.*, a minimal cut-set with $i \in S$ and $j \in T$.

Menger's theorem

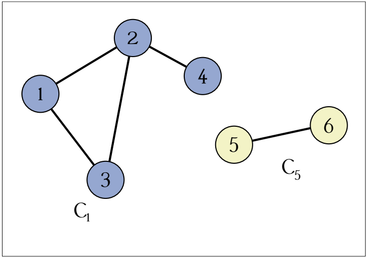
Theorem (Edge-connectivity version)

For an undirected graph G, the size of the minimum edge cut for an arbitrary pair of nodes $i \neq j$ is equal to the maximum number of edge-disjoint paths from i to j.

- Edge-disjoint means they share no common edges
- There is also a node connectivity version
- It also works for digraphs and infinite graphs
- The theorem is generalised in many optimisation algorithms: *e.g.*, maximum flow algorithms.

Connected Components

- A *connected component* is a maximal connected subgraph



- The set of connected components $\{C_i\}$ form a *partition* of the nodes

Definition (Partition)

A *partition* is a set of covering and disjoint subsets, i.e, $\{C_i\}_{i=1}^n$ is a partition of C iff

$$\bigcup_{i=1}^{n} C_i = C \quad \text{and} \quad C_i \cap C_j = \phi, \ \forall i \neq j.$$

Connected Components Algorithm

Data: A Graph $G = (N, E)$

Result: A set of connected components $\{C_i\}$

Initialise $N' = N$;

while $(N' \neq \phi)$ **do**

  Choose a node $i \in N'$ and delete it from N';

  Set $C_i = \{i\}$ and $L = \{(i,j) | (i,j) \in E\}$;

  while $(L \neq \phi)$ **do**

    Choose a link $(k, m) \in L$;

    if $m \notin C_i$ **then**

      add m to C_i;

      delete m from N';

      add all links $(m, l) \in E$ to L;

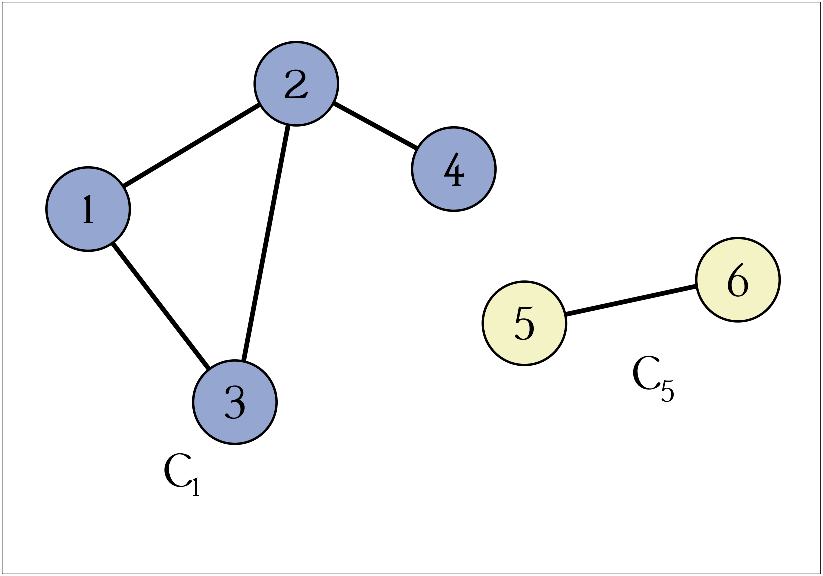
    end

    delete (k, m) `from` L;

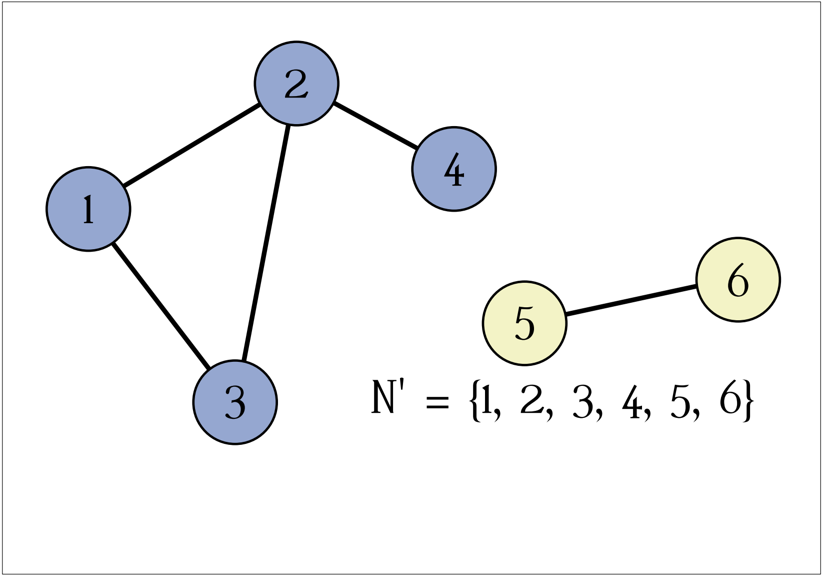
  end

end

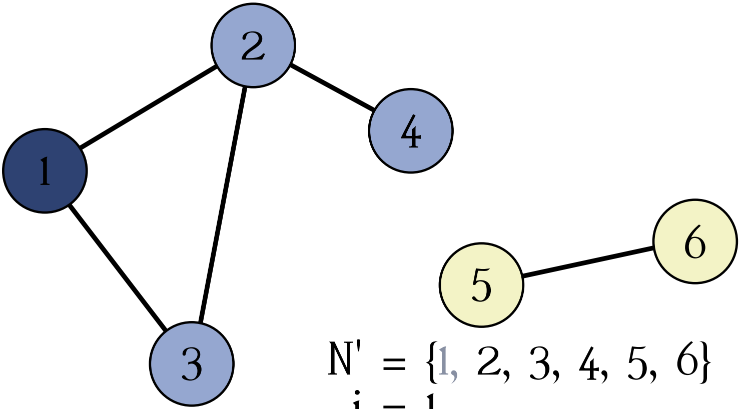
Connected Components Example



Connected Components Example

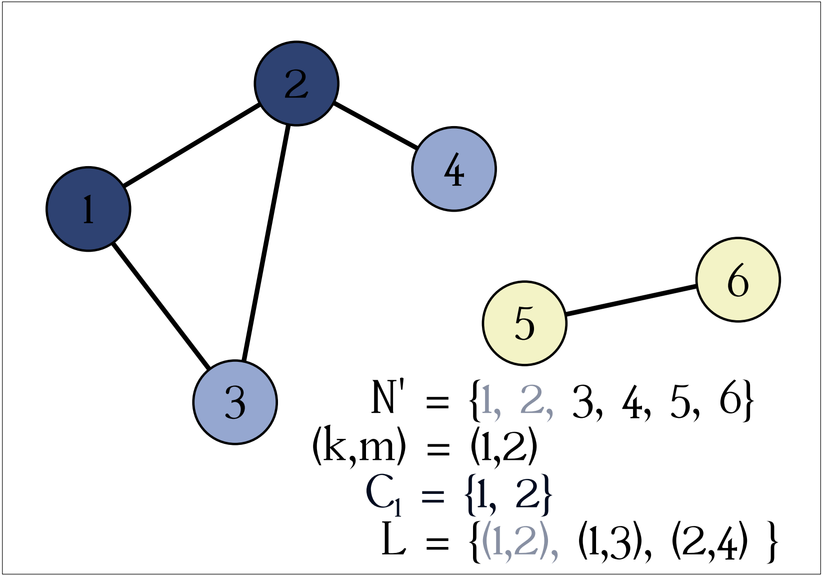


Connected Components Example

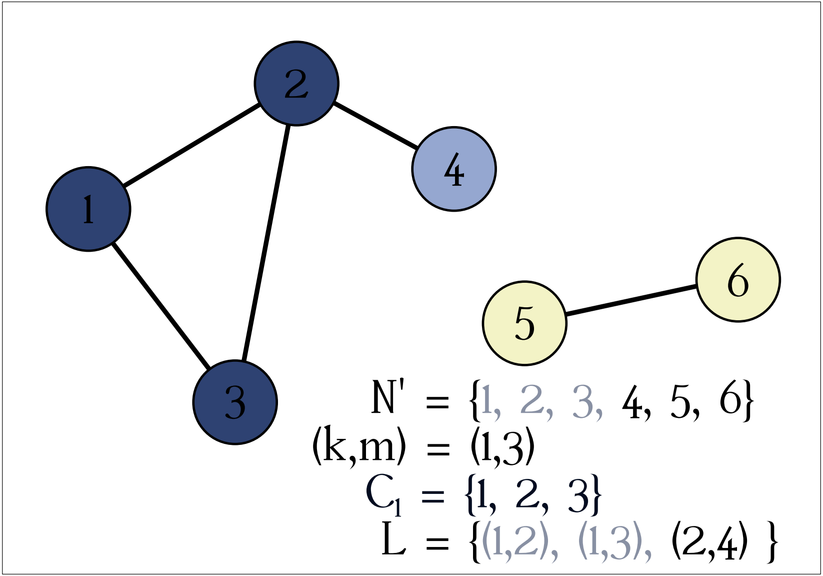


$$N' = \{1, 2, 3, 4, 5, 6\}$$
$$i = 1$$
$$C_1 = \{1\}$$
$$L = \{(1,2), (1,3)\}$$

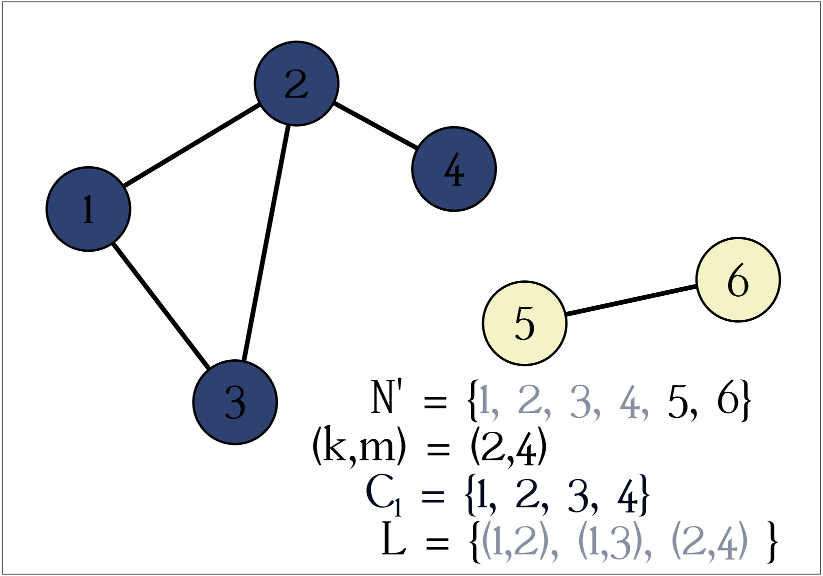
Connected Components Example



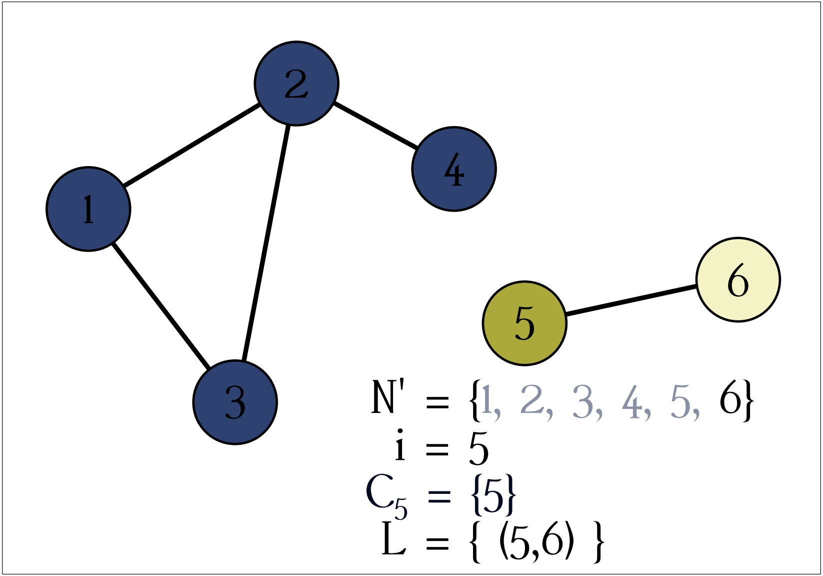
Connected Components Example



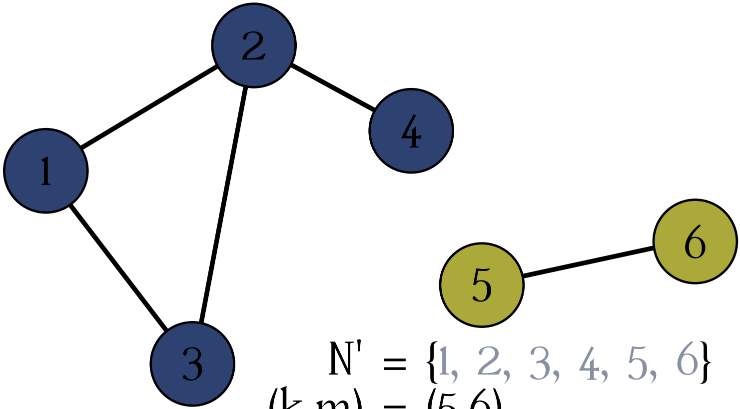
Connected Components Example



Connected Components Example



Connected Components Example



$$
\begin{aligned}
N' &= \{1,\ 2,\ 3,\ 4,\ 5,\ 6\} \\
(k,m) &= (5,6) \\
C_5 &= \{5,\ 6\} \\
L &= \{\ (5,6)\ \}
\end{aligned}
$$

Application 1

Key requirements for critical infrastructure networks (*e.g.*, Internet, Water, Power, ...)

- Reliability

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Application 1

The simplest definition of "reliability" used in networks is some variant of k-connectedness

- The particular variant depends on the failure modes of the network
  - do the nodes fail, or the edges (or both)?
- Leads to network designs with redundancy
  - not necessarily k-fold redundancy
- Is this a good enough definition of reliability

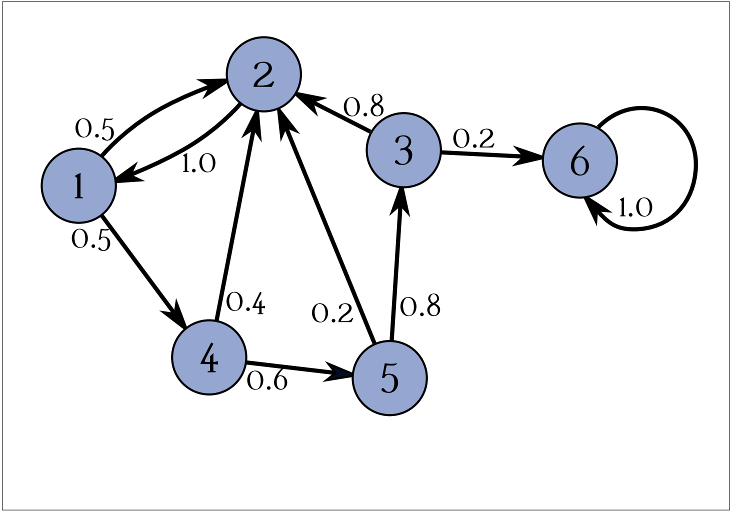
Application 2

Markov chains probability transition matrix

$$P = \begin{pmatrix} 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.8 & 0.4 & 0.2 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.8 & 0.0 \\ 0.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.6 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.2 & 0.0 & 0.0 & 1.0 \end{pmatrix}$$

Application 2

Markov chains are described by a directed graph with self-loops, e.g.,



The transition matrix is just a weighted adjacency matrix.

Application 2

First step of studying a Markov chain is to check its properties

Definition

State i is *accessible* from state j if it is possible to get from i to j.

Accessible = a path from i to j exists

Definition

A Markov chain is *irreducible* if it is possible to get to any state from any state.

Irreducibility = strong connectivity of the graph

Definition

A *communicating class* is a maximal set of mutually accessible states.

Communicating class = connected component

Section 2

Graph Traversal

- Connected components as described above has a little vagueness
  - when I say "choose" how do you choose?
- It's an example of *graph traversal*
  - where we want to visit each node of a graph (at least once)
  - ordered nodes by connectivity
- Traversals used for lots of algorithms
  - could be to search for an element
  - or to calculate a value for each node

  Maybe you don't have the whole graph stored in memory, but have to read bits, e.g., traversing Facebook graph
- There are two main strategies
  - Depth-First Search
  - Breadth-First Search
- For the sake of simplicity, we will assume graphs are connected
- Easiest to understand in neighbour-list representation

Depth-First Search

Visit a neighbour's children before you visit the next neighbour

```
Function DFS(G, i);
Input: A Graph G = (N, E), and start node i ∈ N
label i as explored;
forall j ∈ neighbourhood{i} do
    if j is unexplored then
        DFS(G, j);
    end
end
```

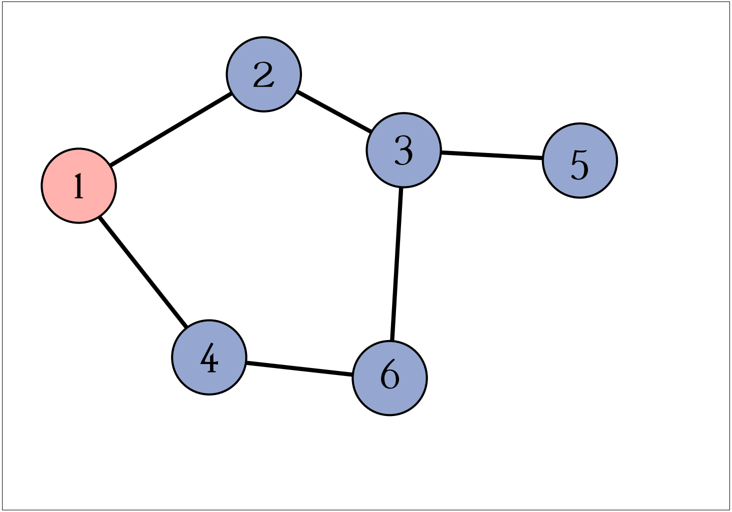
- We could make this faster by avoiding edges going backwards.
- At the moment the algorithm doesn't do anything
  - a *search* also checks something about the node, and returns the first one that checks out
  - but we might also do some sort of update
  - or use to find a connected component ...

Breadth-First Search

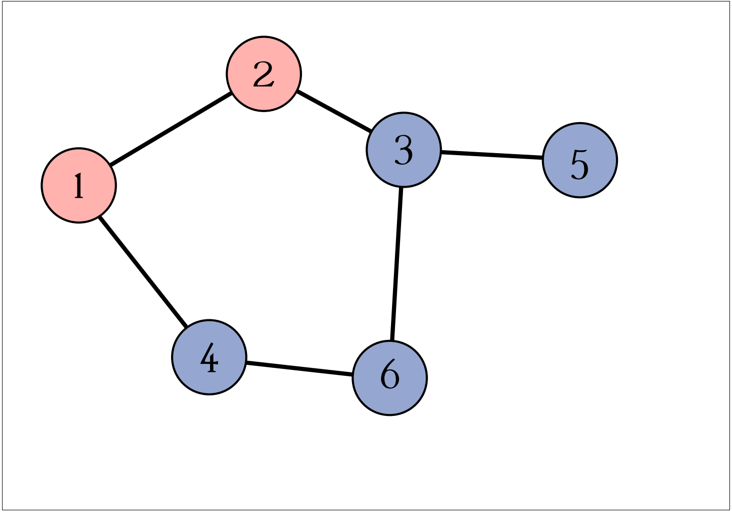
Visit all neighbours before you visit their children

```
Function BFS(G, i);
Input: A Graph G = (N, E), and start node i ∈ N
label i as explored;
create queue Q;
put i on Q;
while Q not empty do
    take j off the front of Q;
    forall k ∈ neighbourhood{j} do
        if k is unexplored then
            label k as explored;
            put k on Q;
        end
    end
end
```

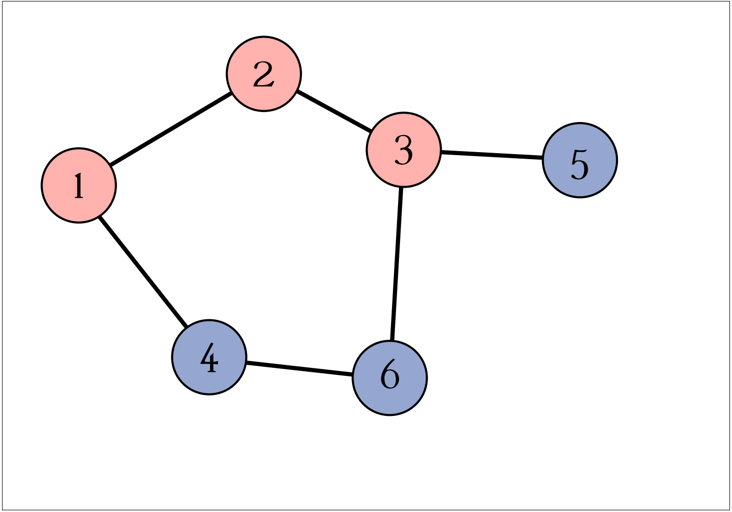
Depth-First Search Example



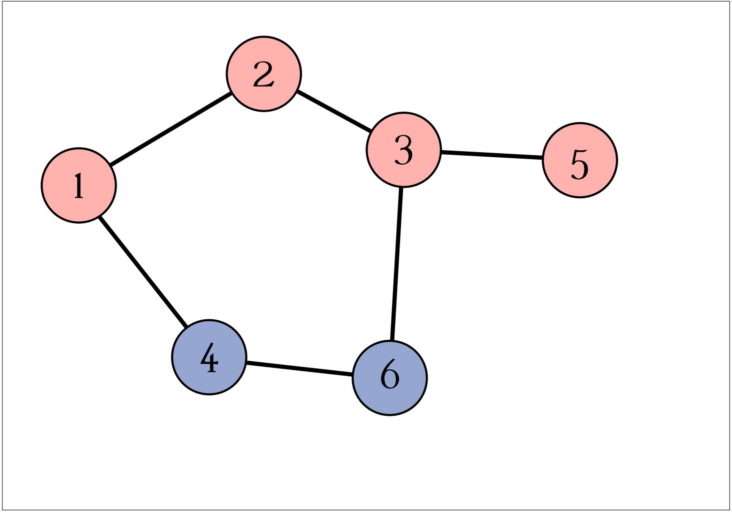
Depth-First Search Example



Depth-First Search Example



Depth-First Search Example



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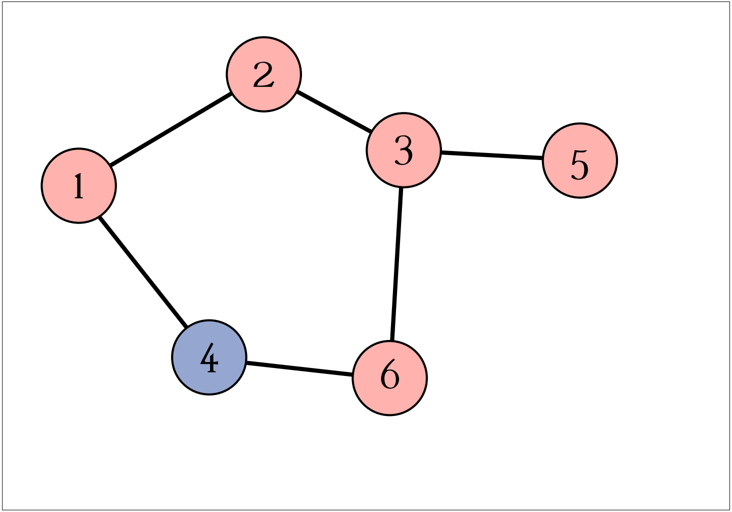

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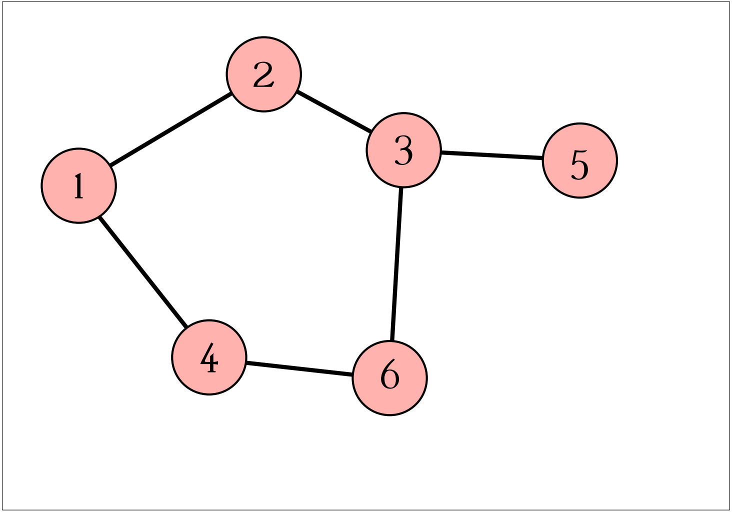

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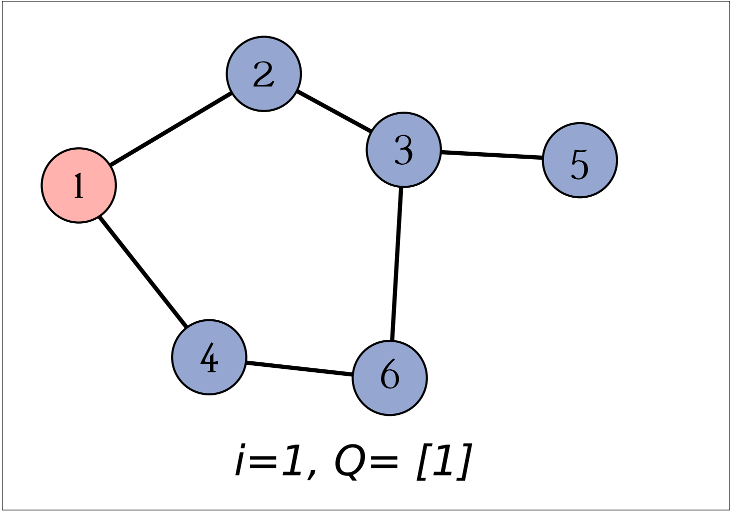

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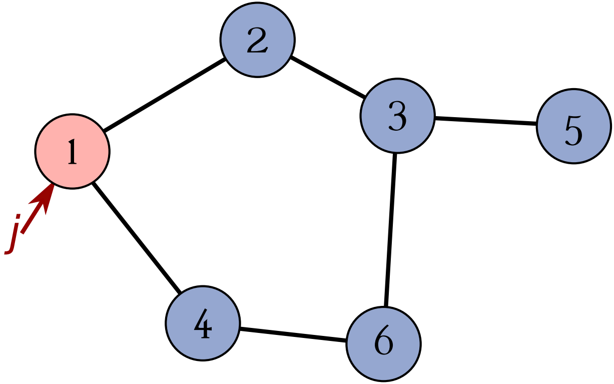
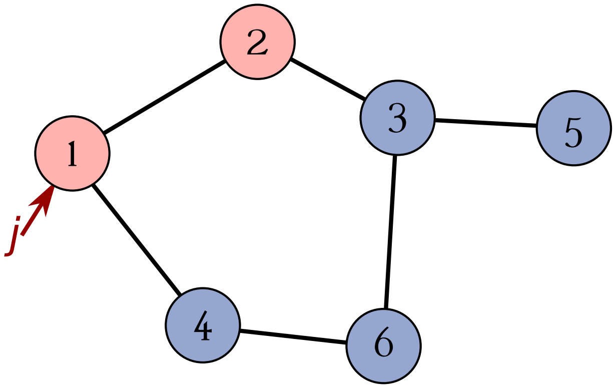


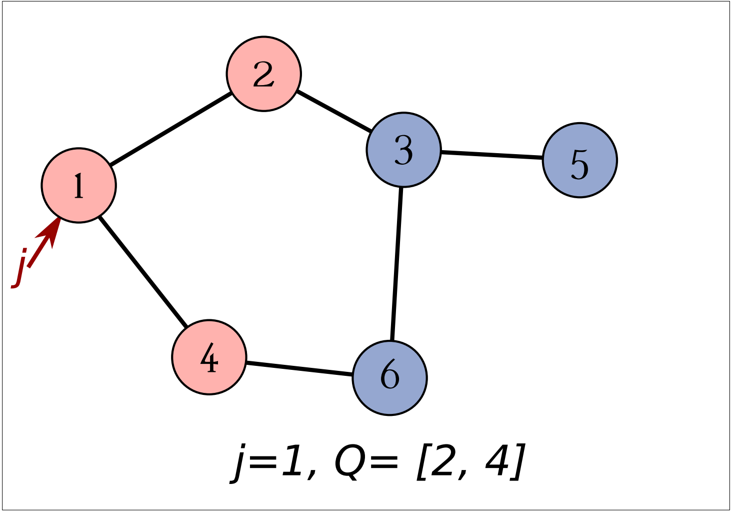
j=1, Q= []

Breadth-First Search Example



j=1, Q= [2]

Breadth-First Search Example



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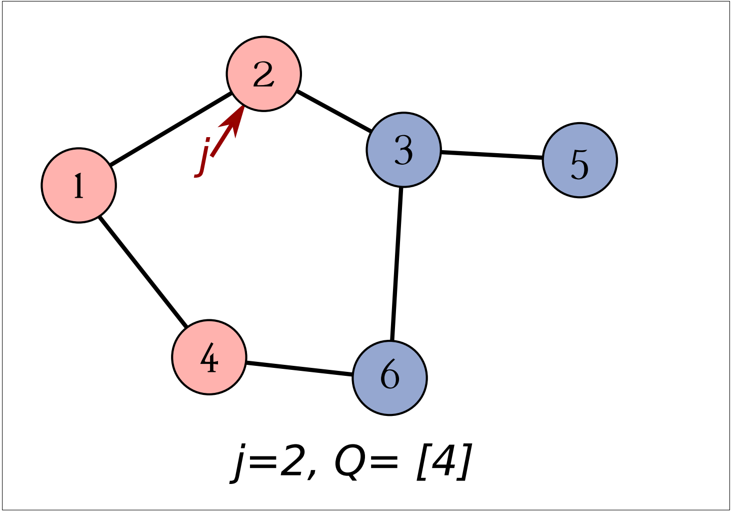

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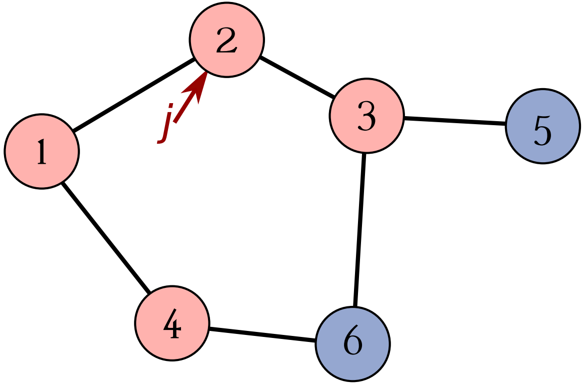
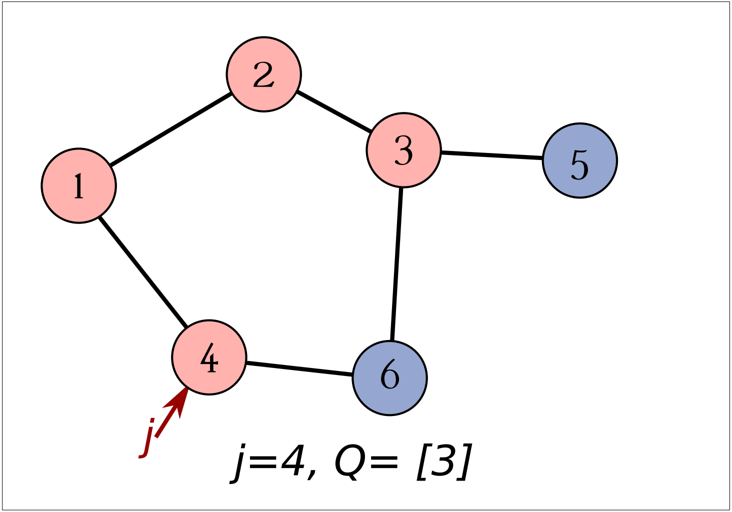
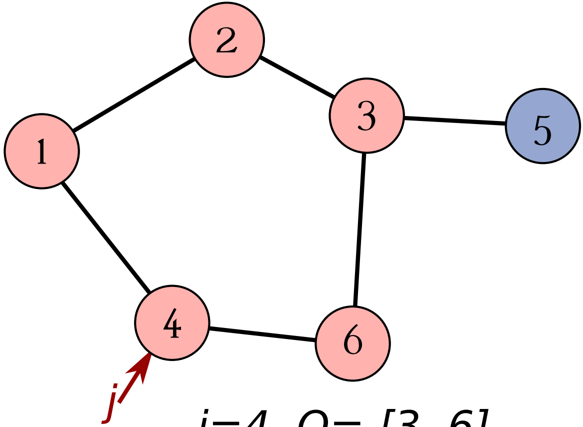


j=2, Q= [4,3]

Breadth-First Search Example

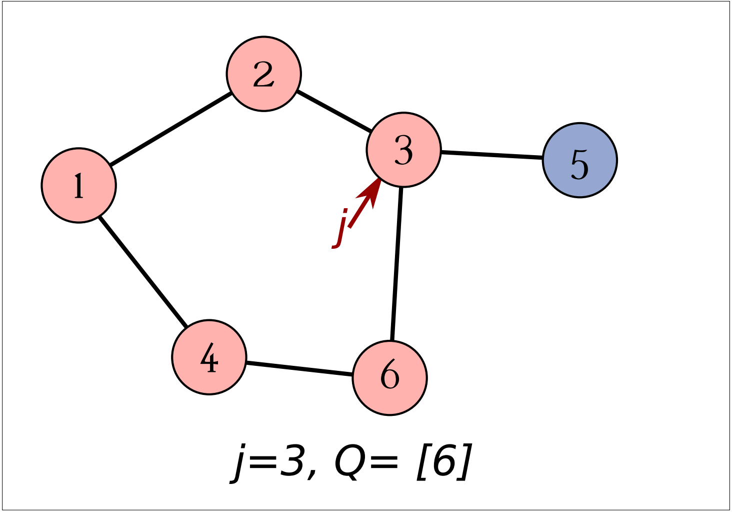


Breadth-First Search Example

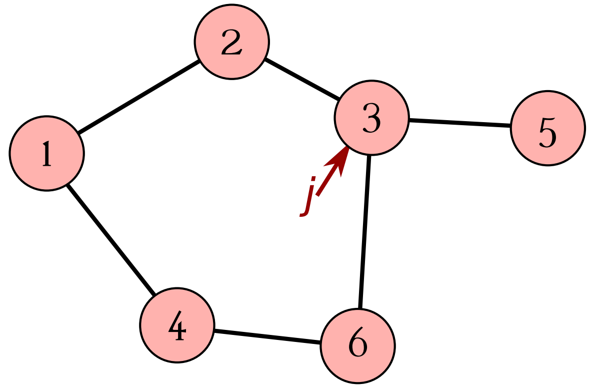


j=4, *Q*= [3, 6]

Breadth-First Search Example

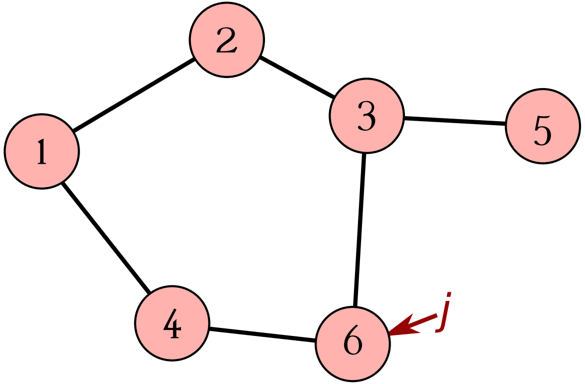


Breadth-First Search Example



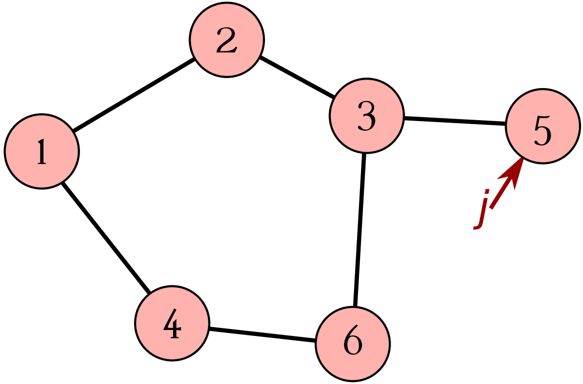
j=3, Q= [6, 5]

Breadth-First Search Example



j=6, Q= [5]

Breadth-First Search Example



j=5, Q= []

Further reading I