

Complex-Network Modelling and Inference

Lecture 16: Operations on graphs (unary operators)

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Operations on graphs

Operations of graphs are important for a number of reasons

- We can use them to build new graph models
- We can calculate properties of graphs
- We use them in proofs of graph properties

Think of them of constructing a grammar or an algebra of graphs.

Operators on graphs

- Types of operators
 - ① operators that calculate properties of graphs (e.g., metrics)
 - ② operators that produce a new graph
 - ③ operators that work on weighted graphs to calculate new weights
- extra notation: for $G = (N, E)$, we define

$$N(G) = N$$

$$E(G) = E$$

i.e., $N(G)$ is the nodes of G and $E(G)$ the edges.

- need to start by defining isomorphic graphs

Graph Isomorphism (reminder)

- First need to know when graphs are the “same”
- Labels often don't matter (or aren't known)
- Two graphs G and H are *isomorphic* if there exists a bijection f between the nodes of G and H

$$f : N(G) \rightarrow N(H)$$

such that it preserves adjacency, *i.e.*,

$$(u, v) \in E(G) \Leftrightarrow (f(u), f(v)) \in E(H)$$

- call the bijection (function) f an isomorphism
- We write two graphs are isomorphic as $G \simeq H$

Section 1

Unary operators

Unary Operators

Operations that map G to G'

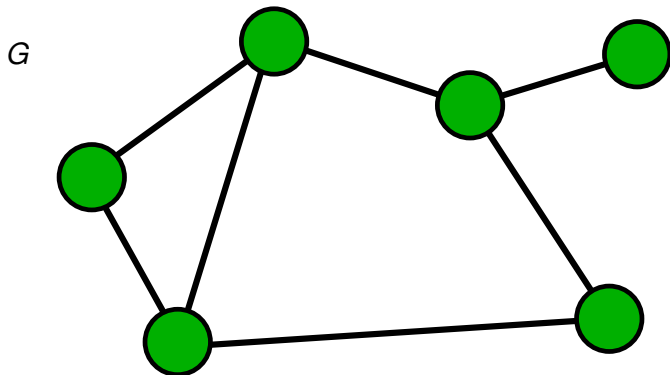
- Complement G^C
- Transpose G^T of a digraph
- Line graph $L(G)$ of graph G
- Power G^k , for $k = 1, 2, \dots$
- Subdivision
- Others
 - ▶ Graph Minor
 - ▶ Mycielskian

Complement G^C

- $N(G^C) = N(G)$ and

$$e \in E(G^C) \Leftrightarrow e \notin E(G)$$

- e.g.,

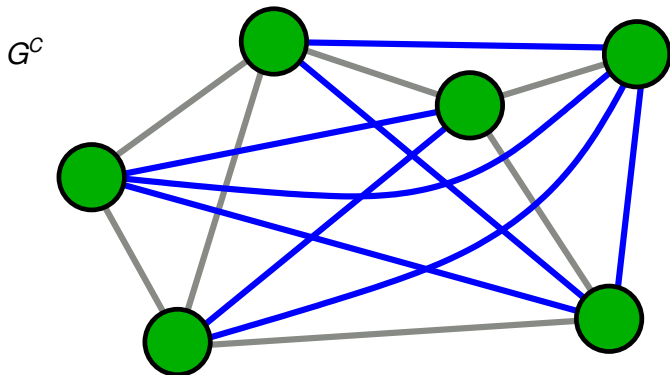


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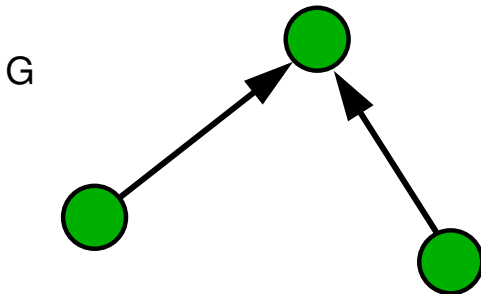
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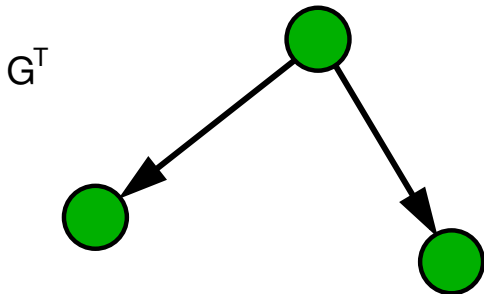
Transpose G^T

- Adjacency matrix is transposed
- Reverse directions of links (in digraph)
- Also called converse, or reverse



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Line graph $L(G)$

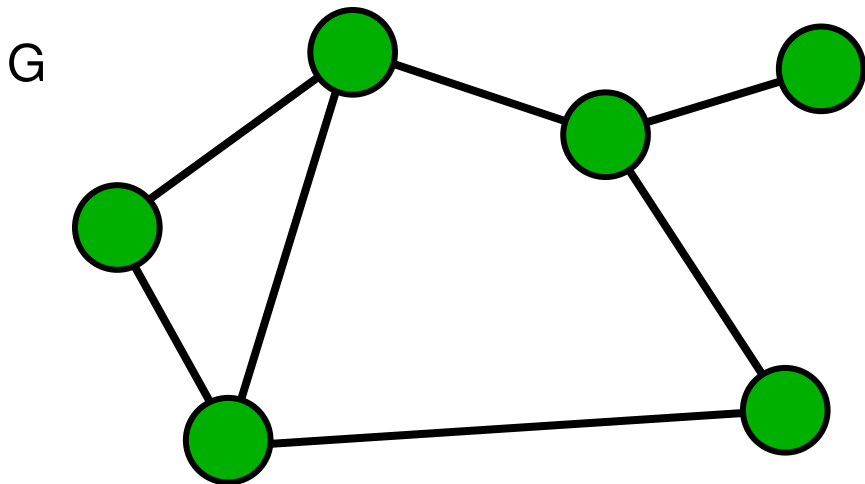
- Sometimes called adjoint, conjugate, edge-to-vertex dual, ...
- Every edge becomes a node
- Node in $L(G)$ adjacent if the corresponding edges in G share a common end-point.
- Formally:

$$G = (N, E) \Rightarrow L(G) = (E, E')$$

where

$$((i, j), (k, m)) \in E' \Leftrightarrow (i = k) \vee (i = m) \vee (j = k) \vee (j = m)$$

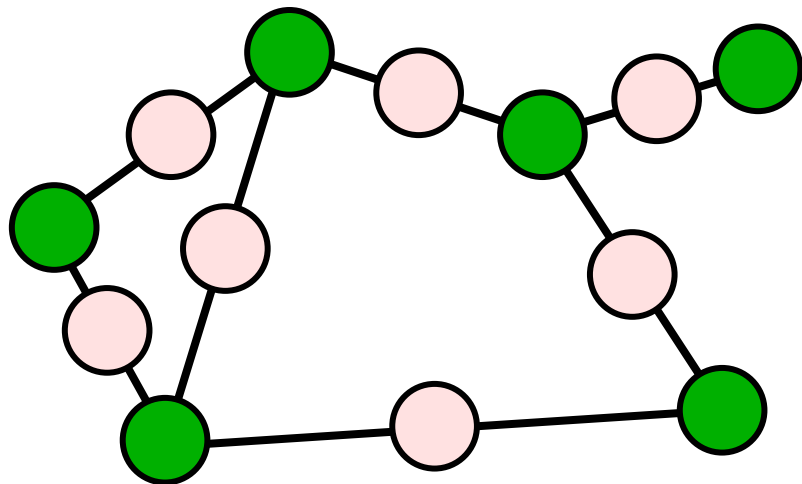
Example Line Graph



Each node in G creates a little clique in $L(G)$.

Example Line Graph

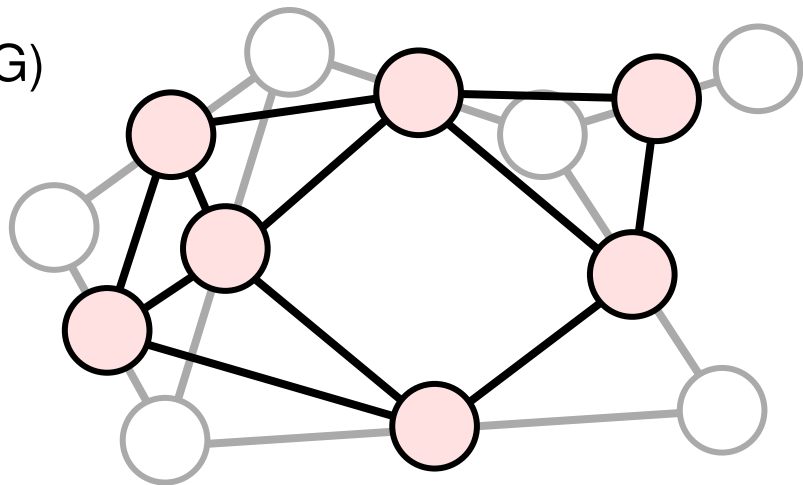
G



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Example Line Graph

$L(G)$



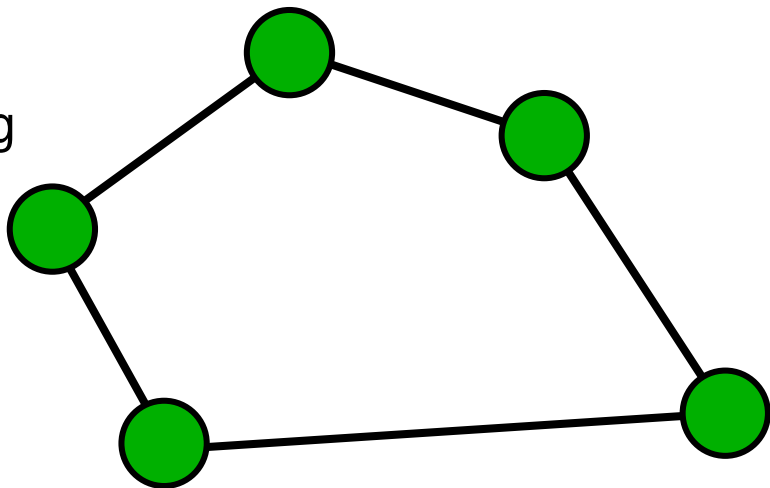
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Properties of Line Graph

- If G is connected, then $L(G)$ is connected
 - ▶ converse is not true
- not all graphs are a line graph
- for a finite connected graph the sequence $G, L(G), L(L(G)), L(L(L(G))), \dots$ has only 4 cases
 - ▶ If G is a cycle graph then they are all isomorphic
 - ▶ If G is a path graph then each subsequent graph is a shorter path until eventually the sequence terminates with an empty graph.
 - ▶ If G is a star with 4 nodes, then all subsequent graphs are triangles
 - ▶ The graphs in the sequence increase indefinitely

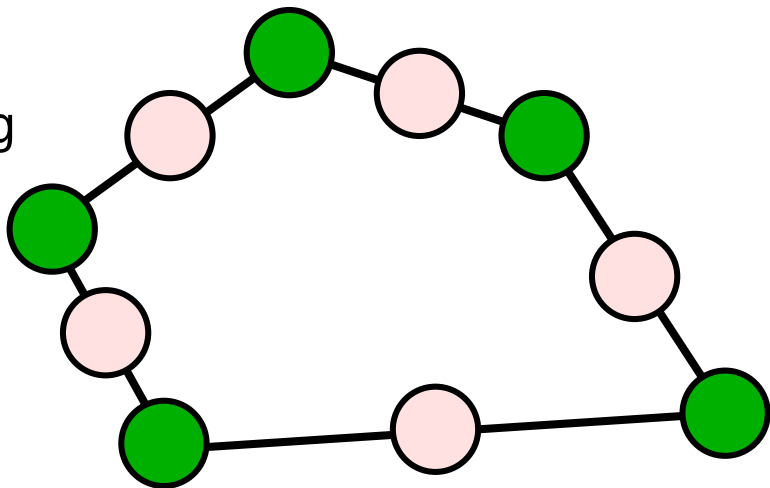
Line Graph: case 1

Ring

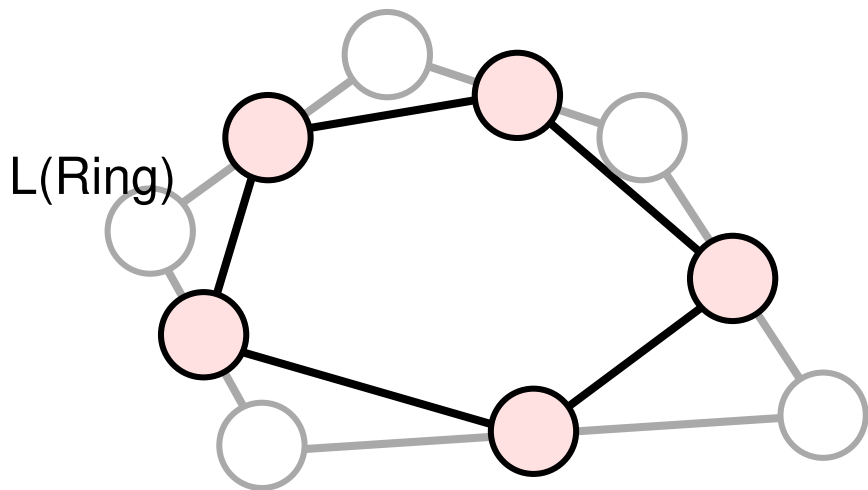


Line Graph: case 1

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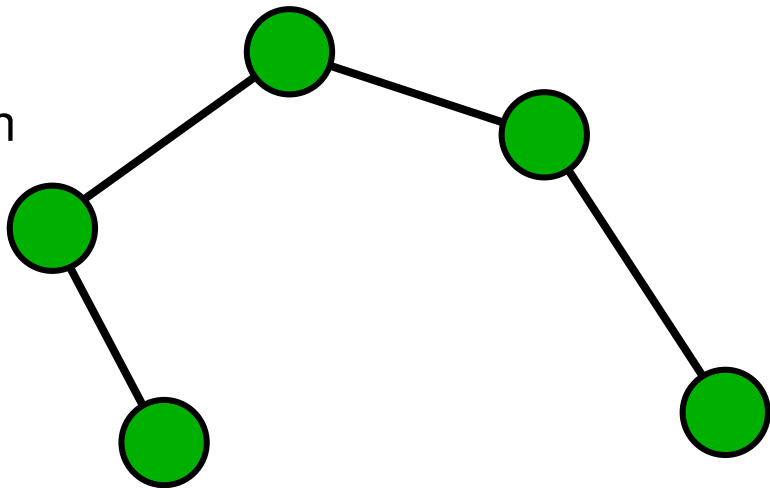


Line Graph: case 1



Line Graph: case 2

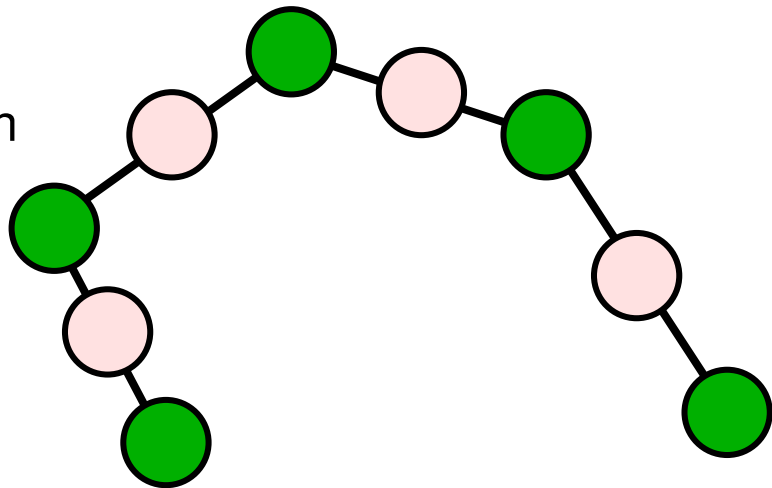
Path



The line graph of a ring is an isomorphic ring (a ring with the same number of nodes).

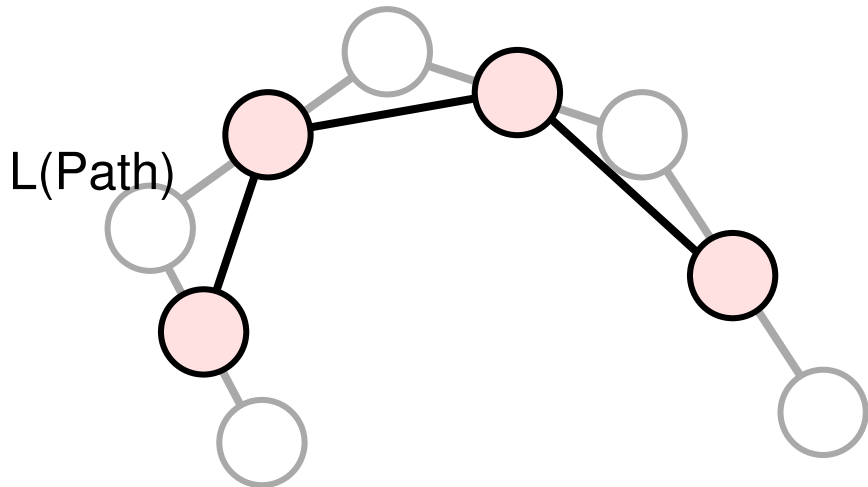
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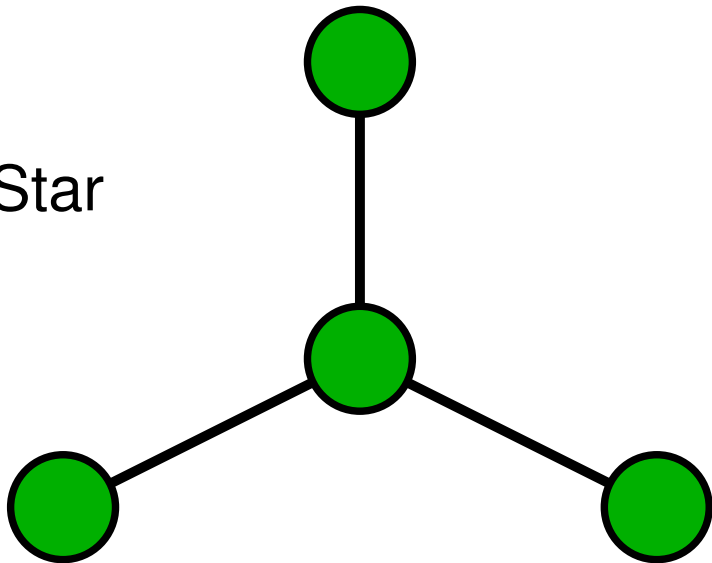
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Line Graph: case 3

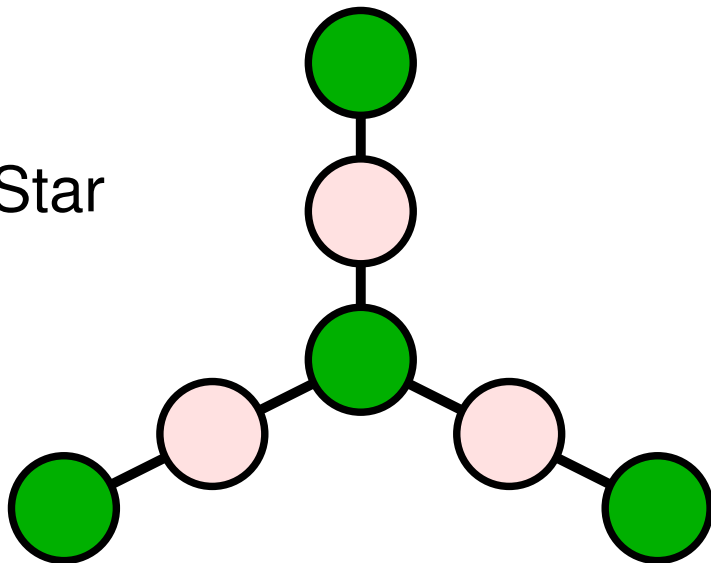
Star



The line graph of a 4 node star is a 3 node ring (a triangle). Using case 1

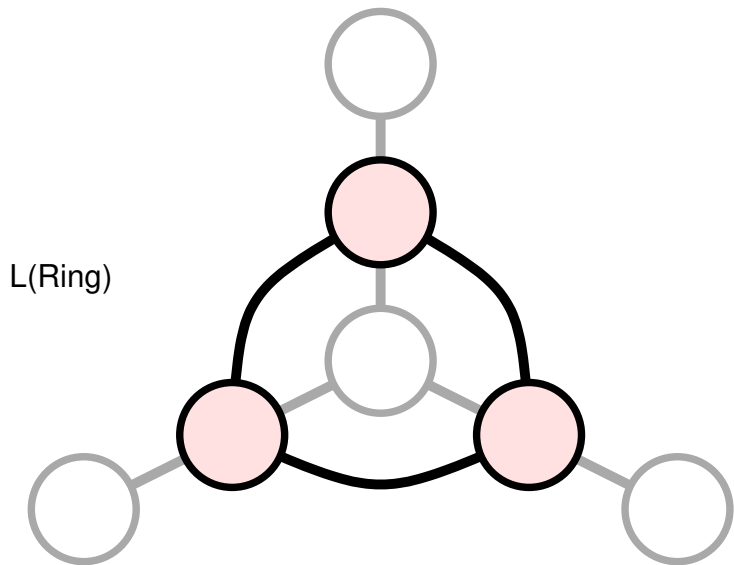
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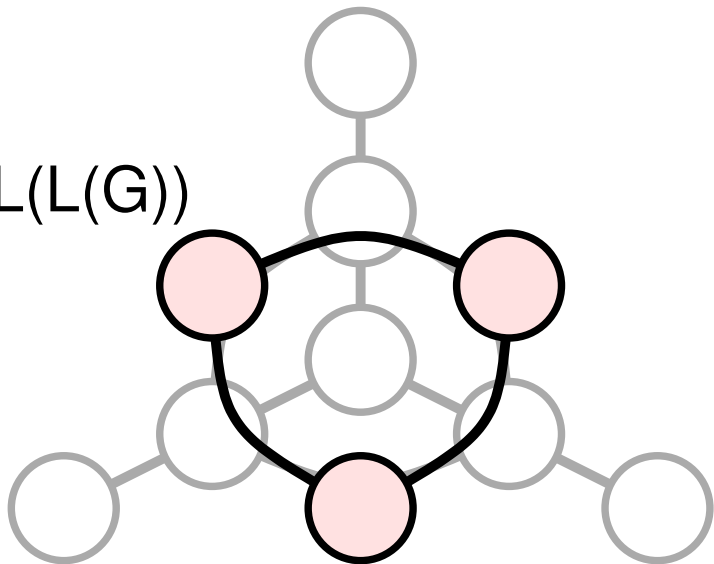
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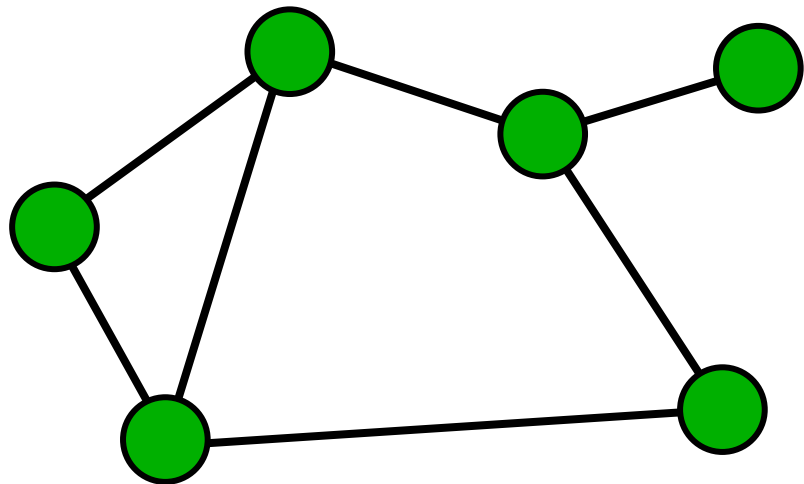
$L(L(G))$



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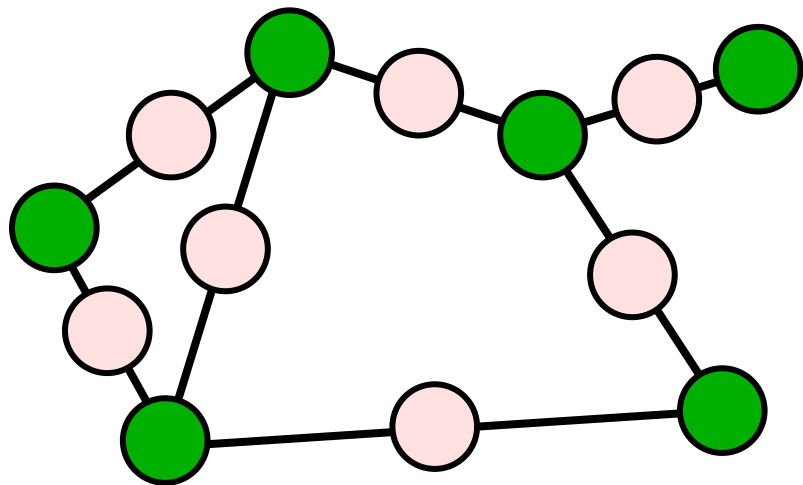
Line Graph: case 4

G



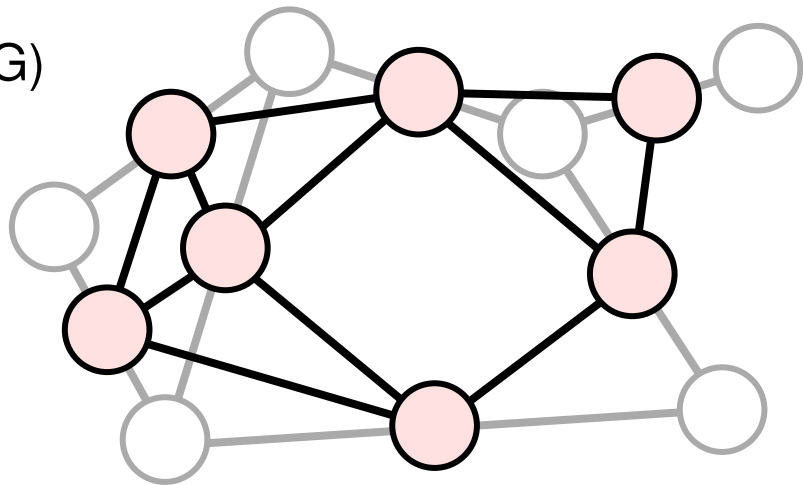
Line Graph: case 4

G



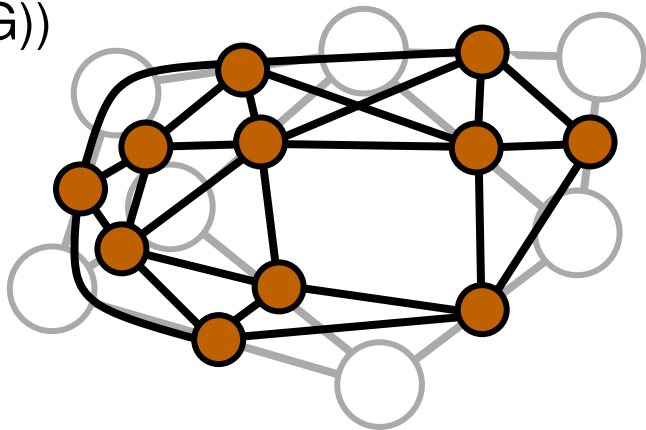
Line Graph: case 4

$L(G)$



Line Graph: case 4

$L(L(G))$



Line Graph growth

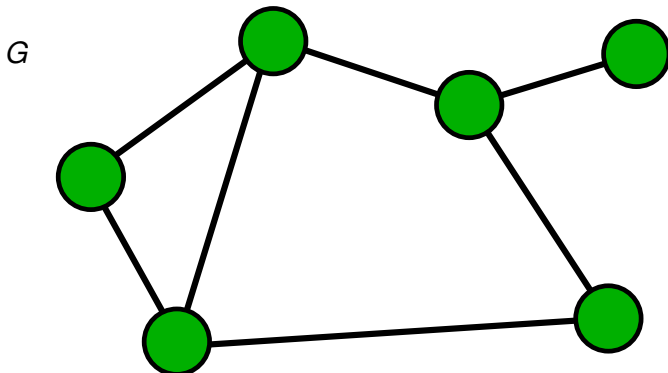
If G has n nodes, and e edges, then $L(G)$ has $n' = e$ nodes and e' edges where

$$e' = \frac{1}{2} \sum_{i=1}^n k_i^2 - e$$

where k_i are the node degrees

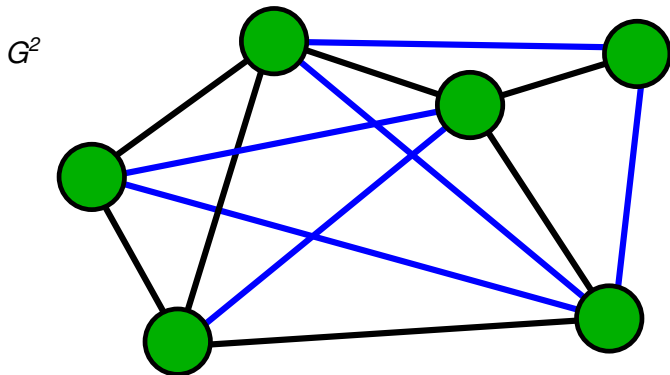
Graph Power G^k

- G^k is the graph formed from the nodes of G , and with edges between all pairs of nodes with (hop) distance no more than k .
- For example:



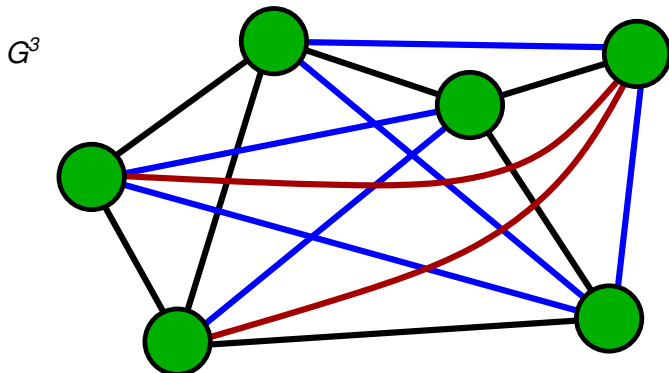
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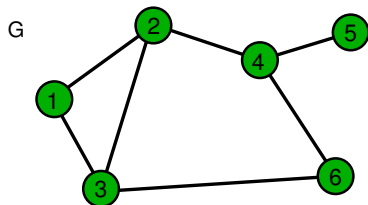
Graph-Power Adjacency Matrix

- We can obtain the adjacency matrix of a graph power G^k , by taking the sum of the first k th powers of the adjacency matrix of G , and thresholding,
- *i.e.*,

$$A^{(k)} = I \left[\left(\sum_{i=1}^k A^i \right) > 0 \right]$$

- ▶ A is the adjacency matrix of a graph power G
 - ▶ $A^{(k)}$ is the adjacency matrix of a graph power G^k
 - ▶ $I(\cdot)$ is an indicator function, applied elementwise to the matrix.
- NB: Element (i, j) in A^k counts the **number** of paths of length k between i and j in the original graph.

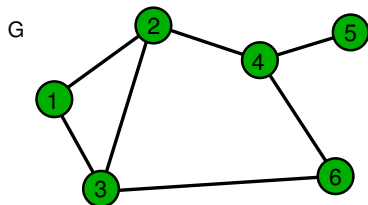
Graph Power G^k example



Adjacency matrix powers

$$A^1 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

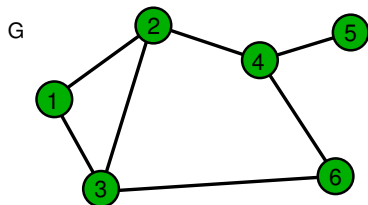
Graph Power G^k example



Adjacency matrix powers

$$A^2 = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 1 \\ 1 & 3 & 1 & 0 & 1 & 2 \\ 1 & 1 & 3 & 2 & 0 & 0 \\ 1 & 0 & 2 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Graph Power G^k example



Adjacency matrix powers

$$A^3 = \begin{pmatrix} 2 & 4 & 4 & 2 & 1 & 2 \\ 4 & 2 & 6 & 6 & 0 & 1 \\ 4 & 6 & 2 & 1 & 2 & 5 \\ 2 & 6 & 1 & 0 & 3 & 5 \\ 1 & 0 & 2 & 3 & 0 & 0 \\ 2 & 1 & 5 & 5 & 0 & 0 \end{pmatrix}$$

Graph-Power Adjacency Matrix

- To understand the above, count the number of a length 2 path between nodes i and j
- Such a path goes through an intermediate node $k \neq i, j$
- Hence the number of length two paths is

$$\begin{aligned} B_{ij} &= \sum_{k \neq i, j} A_{ik} A_{kj} \\ &= \sum_k A_{ik} A_{kj} \quad \text{because } A_{ii} = A_{jj} = 0 \end{aligned}$$

- By definition $B = A^2$
- Induction extends the argument to length k paths.

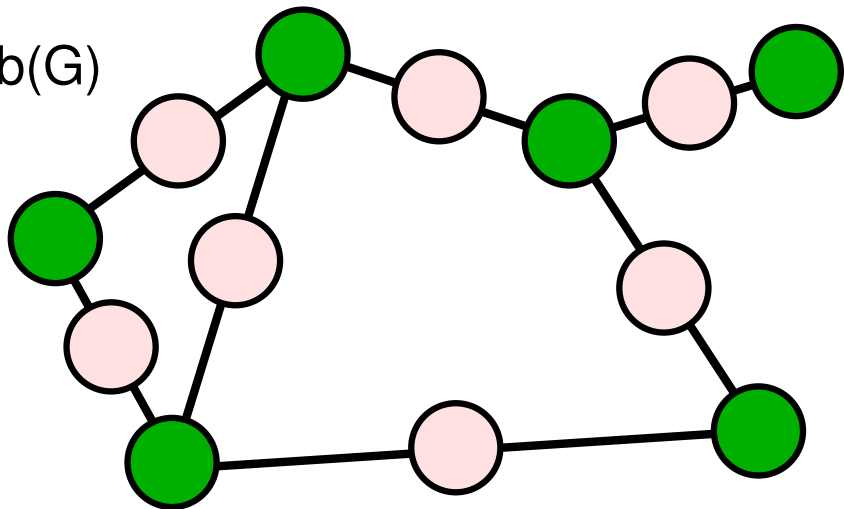
Graph-Power Properties

- For a (strongly) connected (di)graph G with n nodes, is G^n a complete graph (or clique)?
- If the graph has diameter d , then G^d is complete.
- For an unconnected graph, the n th power will be a block-diagonal matrix whose blocks are formed by connected components.
- Square-root graph $G^{1/2}$ is a graph H such that $H^2 = G$.
- NOTE: $G^2 \neq G \times G$
 - ▶ we will talk about multiplication in the next lecture

Subdivision

- Add an extra node into an edge e

sub(G)



Further reading I