Assignment 0: Due **Don't hand in** at 3pm.

Assignments to be handed in through MyUni. Please ensure written assignments are clearly legible. Typed assignments are preferred. Some help may be given in practicals to help get you started with Overleaf/LaTeX in order to present your work well.

This assignment contains revision questions based on what you should have learned in Maths 1, primarily from the Linear Algebra section of the course. You should attempt all questions, and revise your 1st year materials where needed. Ability to work with the tools and concepts in these questions will be assumed in this course.

1. Prove that the intersection of two convex sets $\mathcal{C}_1$ and $\mathcal{C}_2$ is also a convex set.

2. Show whether $f(x) = |x|$ is a convex function or not.

3. (a) What is the definition of linear independence?

   (b) If you have a set of $n+1$ vectors in R^n, can you conclude in general whether those vectors linearly dependent or linearly independent and why?

4. Show that MM^T is a symmetric matrix, for **all** matrices M.

5. Write the following system in augmented matrix form.

$$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 2 \\ x & & & + & z & = & 0 \\ 2x & + & y & + & 3z & = & 2 \end{array}$$

6. Which of the following are linear equations in x_1, x_2 and x_3?
   (a) $x_1 + 4x_2x_3 + 5x_3 = 1$, (b) $3x_1 - x_2 + x_3 = 0$, (c) $x_1 \leq x_2$, (d) $x_1 - x_2 + \sqrt{x_3} = 6$

7. (a) What does it mean when two systems of linear equations are equivalent?

   (b) Find the solution sets for the following two systems. Are they are equivalent?

   (i)
$$\begin{array}{rcr} 2x + 2y & = & 12 \\ x - z & = & 1 \\ y & = & 2 \end{array}$$

   (ii)
$$\begin{array}{rcr} x + 3z & = & 13 \\ -x - 2z & = & -10 \\ x + y - z & = & 3 \end{array}$$

8. In each case, find the elementary row operation that transforms the matrix A_1 into A_2.

   (a)
$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 3 & 7 & 0 \\ 0 & 0 & 1 & 3 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & -2 & -3 \\ 0 & 0 & 1 & 3 & 1 \end{bmatrix}$$

   (b)
$$A_1 = \begin{bmatrix} 1 & 1 & 7 \\ 1 & 3 & 9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 7 \\ 0 & 2 & 2 \end{bmatrix}.$$

9. State whether the following augmented matrices are in *reduced row echelon form.* For those not in reduced row echelon form give a reason why not and state the row operations needed to put the matrix in that form.

$$\text{(a)} \begin{bmatrix} 1 & 0 & 0 & 9/2 & 0 \\ 0 & 1 & 0 & -17/4 & 0 \\ 0 & 0 & 1 & -7/12 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{(b)} \begin{bmatrix} 2 & 0 & 0 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 10 \end{bmatrix} \text{(c)} \begin{bmatrix} 1 & 0 & 0 & 3 & 5 \\ 0 & 1 & 0 & 11 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 6 \end{bmatrix} \text{(d)} \begin{bmatrix} 1 & 0 & 5 & 8 & 5 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

10. Consider the following system of equations $A\mathbf{x} = \mathbf{b}$:

$$\begin{aligned} 2x_2 - 8x_3 &= 20 \\ -3x_1 + 12x_2 - 3x_3 &= -36 \\ 2x_1 - 8x_2 - 6x_3 &= 14 \end{aligned}$$

   (a) Use Gauss-Jordan elimination on the augmented matrix $[A\, I\, \mathbf{b}]$.
   (b) Writing the final matrix as $[A'\, W\, \mathbf{b}']$ (where A reduces to A', and so on), what is A'?
   (c) Verify that $WA = A'$ and $W\mathbf{b} = \mathbf{b}'$. What is W?
   (d) Solve the system of equations.

11. Draw a graph of the region $S = \{(x, y) | x + y \geq 6, 2x - y \geq 0, 5x + 2y \leq 27\}$.
   (a) Is the region convex? bounded? Determine its vertices.
   (b) Use nonnegative slack variables to convert each inequality in S, to an equality. Then, write out the initial tableau for the system, and use it to show that $(0, 0, -6, 0, 27)$ is a basic solution. Is it feasible? Does it correspond to a vertex of S?
   (c) How many basic solutions might there be?
   (d) Determine the basic solutions corresponding to vertices of S.
   (e) At which point in S is (i) $f(x, y) = y - 3x$ maximised? (ii) $g(x, y) = 2x + y$ minimised?

12. A company manufactures and sells two models of lamps, A and B, the profit being \$15 and \$10 respectively per lamp. Assuming that all lamps which are made can be sold, the company wants to know how many of each model it should make. However, they only have two workers available, Asterix and Paul, who can work at most 100 hours and 80 hours per month respectively. Asterix assembles the lamps and takes 20 minutes to assemble each model A lamp and 30 minutes to assemble each model B lamp. Paul paints the lamps and takes 20 minutes to paint each model A lamp and 10 minutes to paint each model B lamp.
   Write this problem as a Linear Program (showing all detailed coefficients).

13. Sketch the following sets in $\mathbf{R}^2$, showing all the vertices.
   (a) $\{(x_1, x_2) \mid 2x_1 + 3x_2 \leq 9,\ 2x_1 - x_2 \geq 2,\ x_1 \geq 0,\ x_2 \geq 0\}$
   (b) $\{(x_1, x_2) \mid 2x_1 + x_2 \geq 2,\ x_1 + 2x_2 \leq 4,\ x_1 \geq 0,\ x_2 \geq 0\}$

14. Let A be an $n \times n$ matrix.
   (a) Define what it means for $\mathbf{x} \in \mathbb{R}$ to be an eigenvector of A with eigenvalue λ.
   (b) Define the characteristic polynomial of A.
   (c) Find all of the eigenvalues of the following matrices.

$$\text{(i)} \quad A_1 = \begin{bmatrix} 0 & 3 \\ 6 & -3 \end{bmatrix} \qquad \text{(ii)} \quad A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

   You can use MATLAB to do your calculations.

15. Calculate
   (a) $\displaystyle\sum_{n=0}^{10} n$
   (b) $\displaystyle\sum_{m=3}^{6} km$, for $k = 2.5$