Transform Methods & Signal Processing
Class Exercise 2: solutions

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Note, questions marked by a (*) are harder than normal questions, and are to be for bonus marks.

1. $\boxed{\text{2 marks}}$ Derive (from the definition of FT) the continuous Fourier transform of the following functions

(a) $f(t) = r(t)$, where $r(t)$, where r is a rectangular pulse of unit width.
Solution: From the definietion

$$\begin{aligned}
\mathcal{F}\{r(t)\} &= \int_{-\infty}^{\infty} r(t)\, e^{-i2\pi st}\, dt \\
&= \int_{-1/2}^{1/2} e^{-i2\pi st}\, dt \\
&= \int_{-1/2}^{1/2} \cos(2\pi st) - i\sin(2\pi st)\, dt
\end{aligned}$$

But $\cos$ is an even function, and $\sin$ is odd, so the sin component of the integral is zero, and we need only compute

$$\begin{aligned}
\mathcal{F}\{r(t)\} &= \int_{-1/2}^{1/2} \cos(2\pi st)\, dt \\
&= \left[\frac{\sin(2\pi st)}{2\pi s}\right]_{-1/2}^{1/2} \\
&= \frac{\sin(\pi s)}{2\pi s} + \frac{\sin(\pi s)}{2\pi s} \\
&= \frac{\sin(\pi s)}{\pi s} \\
&= \text{sinc}(s)
\end{aligned}$$

2. $\boxed{\text{8 marks}}$ Give the continuous Fourier transform of the following functions

(a) $f(t) = Ae^{-\pi(at)^2} e^{-i2\pi s_0 t}$
Solution:
Useful facts

- The FT of a Gaussian is given by a Gaussian, e.g. $\mathcal{F}\left\{e^{-\pi t^2}\right\} = e^{-\pi s^2}$
- Scaling: $\mathcal{F}\{f(at)\} = \frac{1}{|a|}F(s/a)$, so

$$\mathcal{F}\left\{e^{-\pi(at)^2}\right\} = \frac{1}{|a|}e^{-\pi(s/a)^2}$$

- From linearity $\mathcal{F}\{Af\} = A\mathcal{F}\{f\}$, so

$$\mathcal{F}\left\{Ae^{-\pi(at)^2}\right\} = \frac{A}{|a|}e^{-\pi(s/a)^2}$$

- Frequency shift: $\mathcal{F}\left\{f(t)e^{-i2\pi ft}\right\} = F(s+f)$, so the final result is

$$\mathcal{F}\left\{Ae^{-\pi(at)^2} e^{-i2\pi s_0 t}\right\} = \frac{A}{|a|}e^{-\pi((s+s_0)/a)^2}$$

Note that there is a correction to the frequency shift property in the notes (sign change from $-$ to $+$), so no marks were deducted if you used this property.

Note that the time-function is a Gabor function with no time shift applied.

(b) $f(t) = \cos(2\pi s_0 t) * r(t)$, where r is a rectangular pulse, of unit width.
Solution:

- From above we know $\mathcal{F}\{\cos(2\pi s_0 t)\} = (\delta(s-s_0) + \delta(s+s_0))/2$
- The FT of a rectangular pulse is a sinc function, e.g. $\mathcal{F}\{r(t)\} = \text{sinc}(s)$.
- Convolution theorem states that a convolution in the time domain equals a product in the frequency domain, so the result will be

$$\mathcal{F}\{\cos(2\pi s_0 t) * r(t)\} = \frac{\text{sinc}(s)}{2}(\delta(s-s_0) + \delta(s+s_0))$$

(c) $f(t) = \frac{d^2}{dt^2}\text{sinc}(t)$
Solution: The Fourier transform of the sinc function is (by duality) $F(s) = r(-s)$ where r is the rectangular pulse. Note that r is an even function, so $F_{\text{sinc}}(s) = r(s)$.
The Fourier transform of a derivative is given by

$$\mathcal{F}\left\{\frac{d^2}{dt^2}f(t)\right\} = (i2\pi s)^2 F(s)$$

and so the results is

$$\mathcal{F}\left\{\frac{d^2}{dt^2}\text{sinc}(t)\right\} = (i2\pi s)^2 r(s) = -(2\pi s)^2 r(s)$$

(d) $f(x,y) = \exp\left(-\pi(xcos(\theta) + y\sin(\theta))^2\right)$

3*. $\boxed{\text{5 marks}}$ Prove that the continuous Fourier Transform, and Inverse Fourier transform, are really inverse operators for all smooth functions, e.g. show that

$$\mathcal{F}^{-1}\{\mathcal{F}\{f(t)\}\} = f(t)$$

for all functions with at least two continuous derivatives.

[Hint: multiply the signal by a Gaussian, and then relax the Gaussian by increasing its standard deviation, taking the limits. Be careful in taking limits of integrals.]

Solution: Firstly note that I should have included in the question the requirement that the function be continuous, and so we don't need to consider functions which vary on sets of measure zero.

From the definitions

$$\mathcal{F}^{-1}\{\mathcal{F}\{f(t)\}\} = \iint_{-\infty}^{\infty} f(\tau)e^{-i2\pi s\tau}\, d\tau\, e^{i2\pi st}\, ds$$

We start the problem using Fubini's theorem, e.g. if

$$\iint_{-\infty}^{\infty} |h(x,y)|\, dx\, dy < \infty$$

then

$$\iint_{-\infty}^{\infty} h(x,y)\, dx\, dy = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(x,y)\, dx\right]\, dy = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(x,y)\, dy\right]\, dx$$

However, for an arbitrary smooth function, we cannot guarantee the condition for Fubini to be applicable. We get around this by including a Gaussian kernal in the integral, e.g. we consider a new function with Fourier transform $G_\sigma(s)F(s)$, so that

$$I_\sigma(t) = \iint_{-\infty}^{\infty} G_\sigma(s) f(\tau) e^{-i2\pi s\tau}\, d\tau\, e^{i2\pi st}\, ds = \iint_{-\infty}^{\infty} G_\sigma(s) f(\tau) e^{-i2\pi s(\tau-t)}\, d\tau\, ds$$

where

$$G_\sigma(s) = e^{-s^2/2\sigma^2}$$

which makes the integrals finite. Note that this is an *unnormalized Gaussian*, i.e., we ommit the factor $\frac{1}{\sqrt{2\pi\sigma^2}}$ that would lead to $\int_{-\infty}^{\infty} G_\sigma(s) ds = 1$.

Now, we can compute this double integral in two different ways (because it satisfies the condition of Fubini's theorem), to get firstly

$$\begin{aligned} I_\sigma(t) &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) e^{-i2\pi s\tau}\, d\tau \right] G_\sigma(s) e^{i2\pi st}\, ds \\ &= \int_{-\infty}^{\infty} F(s) G_\sigma(s) e^{i2\pi st}\, ds \end{aligned}$$

Computing the integral the other way we get

$$\begin{aligned} I_\sigma(t) &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} G_\sigma(s) e^{i2\pi s(t-\tau)}\, ds \right] f(\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} g_\sigma(t-\tau) f(\tau)\, d\tau \end{aligned}$$

where we note that the inverse Fourier transform $\mathcal{F}^{-1}\{G_\sigma(s)\} = g_\sigma(t) = \sqrt{2\sigma^2/\pi} e^{-2\sigma^2 s^2}$. Note that this Gaussian gets taller and narrower as G_σ gets flatter (i.e., as $\sigma \to \infty$. Also $g_\sigma(t)$ is normalized, i.e., $\int_{-\infty}^{\infty} g_\sigma(s) ds = 1$ for all σ. In the limit as $\sigma \to \infty$, we get $g_\sigma(s) \to \delta(t)$.

The Dominated Convergence Theorem relates states that, given a family of functions $\{f_n\}$ such that

$$\lim_{n\to\infty} f_n(t) = f(t)$$

almost everywhere, if for all n

$$|f_n(t)| \leq g(t)$$

for a function $g(t)$ such that $\int_{-\infty}^{\infty} g(t)\, dt < \infty$ then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(t)\, dt = \int_{-\infty}^{\infty} f(t)\, dt$$

In our case, the series of functions could be given by $f_\sigma(t)$ for $\sigma = n$. Firstly note that if the function $f \in L^1(\mathbb{R})$ then

$$|F(s)| = |\mathcal{F}\{f(t)\}| = \left| \int_{-\infty}^{\infty} f(\tau) e^{-i2\pi s\tau}\, d\tau \right| \leq \int_{-\infty}^{\infty} |f(\tau) e^{-i2\pi s\tau}|\, d\tau = \int_{-\infty}^{\infty} |f(\tau)|\, d\tau < \infty$$

So for L^1 functions, the magnitude of the Fourier transform is finite. Hence $F(s)G_\sigma(s)$ is dominated by $F_{\max}$, and so we can apply the dominated convergence theorem (noting that $\lim_{\sigma\to\infty} G_\sigma = 1$) to obtain

$$\lim_{\sigma\to\infty} I_\sigma(t) = \int_{-\infty}^{\infty} F(s) e^{i2\pi st}\, ds$$

which is just the inverse Fourier transform of the Fourier transform, i.e., $\mathcal{F}^{-1}\{\mathcal{F}\{f\}\}$.

In the second case, we have (inside the integral) $g_\sigma(t-\tau)f(\tau)$. Note that $g_\sigma(0) = \sqrt{2\sigma^2/\pi}$, which increases with σ, so we don't have a simple dominated function here. But the σ term is constant with respect to the integral, and so we can pull it outside the integral, and simply consider the integral over $e^{-2\sigma^2 s^2} f(t)$, which is dominated by $|f(t)|$ (which must be finite because it is in L^1 and continuous). Hence, the integral converges to give

$$\int_{-\infty}^{\infty} \frac{\delta(t-\tau)}{\sigma} f(\tau)\, d\tau$$

and the factors of σ cancel to give

$$\lim_{\sigma\to\infty} I_\sigma(t) = \int_{-\infty}^{\infty} \delta(t-\tau) f(\tau)\, d\tau = f(t)$$

Hence

$$\mathcal{F}^{-1}\{\mathcal{F}\{f\}\} = f(t)$$