Variational Methods and Optimal Control
Class Exercise 2: due before lecture, on Thursday 23rd August, 2012

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1. For the fixed end point problem to find the extremals of a functional

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y')\, dx$$

where f has continuous partial derivatives of up to second order wrt x, y and y', state the Euler-Lagrange equation that the extremal curve must statisfy.

2. Find the extremals of the functionals

(a) $F\{y\} = \int_a^b \frac{\sqrt{1+y'^2}}{y}\, dx$

(b) $F\{y\} = \int_0^1 \left[xy^2 + (y + x^2 y)y'\right] dx$, subject to $y(0) = 0$, and $y(1) = 2$.

(c) $F\{y\} = \int_0^1 \left[xy^2 + (y + xy^2)y'\right] dx$, subject to $y(0) = 0$, and $y(1) = 2$.

or give reasons why no extremal exists.

3. A functional F is given by

$$F[y] \; = \; \int_0^1 x(1-x) y y''\, dx.$$

Use an appropriate integration by parts to show that F can be expressed in the standard form

$$F[y] \; = \; \int_0^1 f(x, y, y')\, dx$$

and derive an ordinary differential equation that must be satisfied by any extremal to F.

4. State if the following functionals are or are not autonomous, degenerate, and/or have dependence on y.

(a) $F\{y\} = \int_a^b \frac{\sqrt{1+y'^2}}{y}\, dx$

(b) $F\{y\} = \int_a^b y^2 y' + xy'\, dx$

(c) $F\{y\} = \int_a^b \cos(xy') + \sin(xy')\, dx$

(d) $F\{y\} = \int_a^b \cos^2(y') + \sin^2(xy)\, dx$

Please provide your answer in the form of a table whose rows correspond to each integral, and which has a column for each case. Fill in all parts of the table.

5. Find the shape of a geodesic on the (curved part) of the surface of a cylinder.

Can you explain the geodesic by “unrolling” the cylinder?