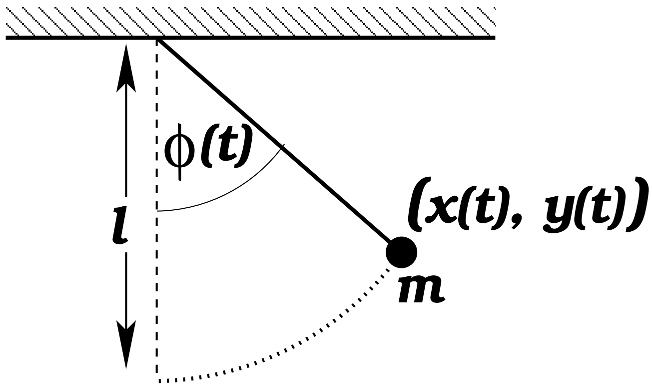
Variational Methods and Optimal Control
Class Exercise 6: do not hand in

Matthew Roughan
<matthew.roughan@adelaide.edu.au>

1: **Conservation laws:** use Neother's theorem to relate the symmetries of the pendulum to the conservation laws that apply to the system. More specifically, consider the system as follows:



Kinetic energy

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}ml^2\dot{\phi}^2$$

Potential energy

$$V = mg(l - y) = mgl(1 - \cos\phi)$$

The Lagrangian is

$$L(\phi, \dot{\phi}) = \frac{1}{2}ml^2\dot{\phi}^2 - mgl(1 - \cos\phi),$$

and the action integral is

$$F\{\phi\} = \int_{t_0}^{t_1} \left(\frac{1}{2}ml^2\dot{\phi}^2 - mgl(1 - \cos\phi) \right) dt.$$

Determine whether the Lagrangian has translation (in space or time) or rotation invariance, and thence determine the conservation laws that apply.

2. Broken extremals: Minimize the functional

$$F\{x\} = \int_0^2 (\dot{x} + 1)^2 \dot{x}^2 \, dt$$

subject to the end-point conditions that $x(0) = 1$ and $x(2) = 0$. *[Hint: consider the possibility of broken extremals.]*

3. Optimal control: Express the following in a form of an optimal control problem to which the Pontryagin Maximum Principle can be applied:

(a) Minimize

$$F\{x\} = \int_0^{10} x^2 \, dt$$

subject to

$$|\ddot{x}| \leq 1, \text{ and } x(0) = 1$$

(b) Minimize T subject to

$$\int_0^T \ddot{x}^2 \, dt = 4$$

and

$$x(0) = 1, \text{ and } \dot{x}(0) = 1, \text{ and } \dot{x}(T) = -2$$

4. Optimal control: A person is considering a lifetime plan of investment and expenditure. With initial savings S and no other income other than from an investment with a fixed interest rate $\alpha > 0$, this investor's capital weath at time t is $x(t)$ and is governed by

$$\dot{x} = \alpha x - r$$

where $r = r(t)$ is the investors rate of expenditure. The immediate enjoyment due to expenditure at rate $r(t)$ results in utility $U(r)$, which we will take to be $U(r) = \sqrt{r}$. Future enjoyment at time t is discounted by $e^{-\beta t}$. Thus our investor wishes to maximize

$$J\{r\} = \int_0^T e^{-\beta t} U(r)\, dt$$

subject to $\dot{x} = \alpha x - r$, and the initial condition $x(0) = 1$. Also, at the final time, any remaining capital is wasted, so let $x(T) = 0$. There are additional implicit constraints: we cannot borrow, so capital cannot become negative, and we cannot expend a negative amount, so $r(t) \geq 0$ for all t.

Use the Pontryagin Maximum Principle to find the optimal expenditure strategy $r(t)$.