Variational Methods & Optimal Control

lecture 07

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Special case 3

When f has no explicit dependence on y the E-L equations simplify to give

$$\frac{\partial f}{\partial y'} = const$$

An example where we might use this is in calculating geodesics on non-planar objects such as the sphere.

Euler-Lagrange equation

Theorem 2.2.1: Let $F : C^2[x_0, x_1] \to I\!R$ be a functional of the form

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y')\, dx,$$

where f has continuous partial derivatives of second order with respect to x, y, and y', and $x_0 < x_1$. Let

$$S = \left\{ y \in C^2[x_0, x_1] \,\middle|\, y(x_0) = y_0 \text{ and } y(x_1) = y_1 \right\},$$

where y_0 and y_1 are real numbers. If $y \in S$ is an extremal for F, then for all $x \in [x_0, x_1]$

$$\boxed{\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0}$$

No explicit y dependence

Suppose the function is of the form

$$J\{y\} = \int_{x_0}^{x_1} f(x, y')dx$$

where y does not appear explicitly.

The Euler-Lagrange equation reduces to

$$\frac{\partial f}{\partial y'} = c_1$$

where c_1 is a constant.

Solving

$\frac{\partial f}{\partial y'}$ is a known function of x and y',
so this is a first order DE for y.
In principle for $\frac{\partial^2 f}{\partial y'^2} \neq 0$ can recast

$$\frac{\partial f}{\partial y'} = c_1 \quad \text{as} \quad y' = g(x, c_1)$$

for some g.

Geodesics on the unit sphere

Find the shortest path between two points on the unit sphere.

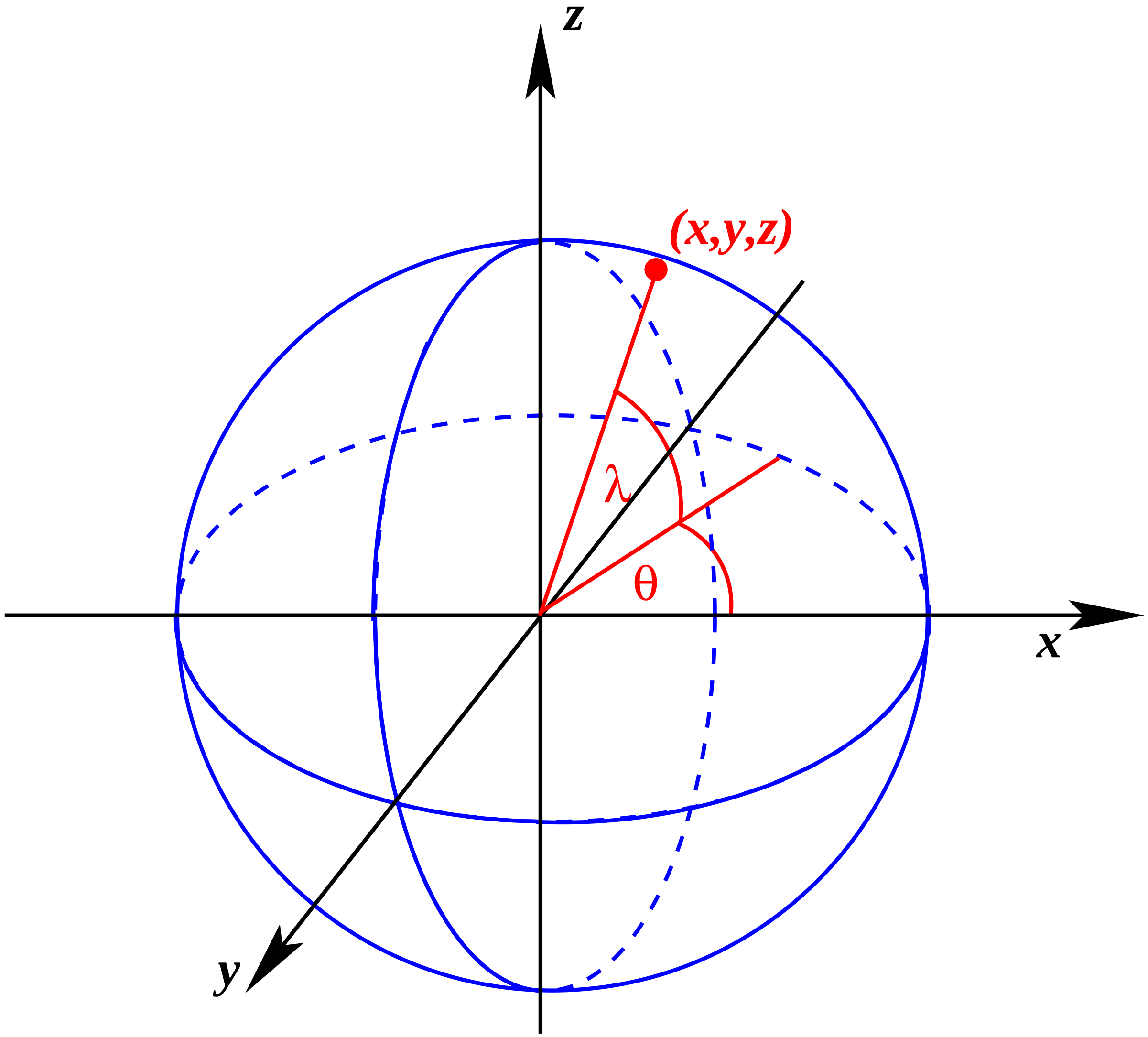
Spherical co-ordinates

efine

$$\begin{aligned} \lambda &= \text{latitude} \\ \theta &= \text{longitude} \end{aligned}$$

artesian co-ordinates (x, y, z)

$$\begin{aligned} x &= \cos(\theta)\cos(\lambda) \\ y &= \sin(\theta)\cos(\lambda) \\ z &= \sin(\lambda) \end{aligned}$$



Transformation to spherical co-ord.

$$
\begin{aligned}
x &= \cos(\theta)\cos(\lambda) \\
y &= \sin(\theta)\cos(\lambda) \\
z &= \sin(\lambda)
\end{aligned}
$$

Chain rule

$$
\begin{aligned}
dx &= \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \lambda} d\lambda &= -\sin(\theta)\cos(\lambda) d\theta - \cos(\theta)\sin(\lambda) d\lambda \\
dy &= \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \lambda} d\lambda &= \cos(\theta)\cos(\lambda) d\theta - \sin(\theta)\sin(\lambda) d\lambda \\
dz &= \frac{\partial z}{\partial \theta} d\theta + \frac{\partial z}{\partial \lambda} d\lambda &= \cos(\lambda) d\lambda
\end{aligned}
$$

$$
ds^2 = dx^2 + dy^2 + dz^2 = d\lambda^2 + \cos^2(\lambda) d\theta^2
$$

Geodesics on the unit sphere

$$\int_{(x(s_0),y(s_0),z(s_0))}^{(x(s_1),y(s_1),z(s_1))} 1ds = \int_{\lambda_0}^{\lambda_1} \left[1+\cos^2(\lambda)\left(\frac{d\theta}{d\lambda}\right)^2\right]^{\frac{1}{2}} d\lambda$$

θ is like y, λ is like x, $\frac{d\theta}{d\lambda} = \theta'$ is like y', hence EL eqn:

$$\begin{aligned}
\frac{\partial}{\partial\theta'}\left[1+\cos^2(\lambda)\theta'^2\right]^{\frac{1}{2}} &= c_1 \\
\frac{\cos^2(\lambda)\theta'}{\left[1+\cos^2(\lambda)\theta'^2\right]^{\frac{1}{2}}} &= c_1 \\
\frac{\cos^4(\lambda)\theta'^2}{1+\cos^2(\lambda)\theta'^2} &= c_1^2
\end{aligned}$$

The constant

$$\frac{\cos^4(\lambda)\theta'^2}{1+\cos^2(\lambda)\theta'^2} = c_1^2$$

Now

$$\theta'^2\cos^4(\lambda) \leq \theta'^2\cos^2(\lambda) \leq 1+\theta'^2\cos^2(\lambda)$$

So

$$c_1 \in [-1,1]$$

So we can replace c_1 with

$$c_1 = \cos(\alpha)$$

Geodesics on the unit sphere

Re-arrange

$$\cos^4(\lambda)\theta'^2 = c_1^2\left(1+\cos^2(\lambda)\theta'^2\right)$$

Re-arrange some more

$$\begin{aligned}\theta'^2 &= \frac{c_1^2}{\cos^4(\lambda)-c_1^2\cos^2(\lambda)}\\ \theta' &= \left\{\frac{c_1^2}{\cos^2(\lambda)(\cos^2(\lambda)-c_1^2)}\right\}^{\frac{1}{2}}\\ \theta' &= g(\lambda,c_1)\end{aligned}$$

Analogous to $y' = g(x,c_1)$.

Solving the DE

Insert $c_1 = \cos(\alpha)$

$$
\begin{aligned}
\theta' &= \frac{\cos(\alpha)}{\cos(\lambda)\left[\cos^2(\lambda) - \cos^2(\alpha)\right]^{\frac{1}{2}}} \\
\theta &= \int \frac{\cos(\alpha)}{\cos(\lambda)\left[\cos^2(\lambda) - \cos^2(\alpha)\right]^{\frac{1}{2}}}\, d\lambda \\
&= \int \frac{\sec^2(\lambda)}{\left[\sec^2(\alpha) - \sec^2(\lambda)\right]^{\frac{1}{2}}}\, d\lambda \\
&= \int \frac{\sec^2(\lambda)}{\left[\tan^2(\alpha) - \tan^2(\lambda)\right]^{\frac{1}{2}}}\, d\lambda \qquad \text{as } \sec^2 x = 1 + \tan^2 x
\end{aligned}
$$

Solving the DE (part ii)

$$\theta = \int \frac{\sec^2(\lambda)}{[\tan^2(\alpha) - \tan^2(\lambda)]^{\frac{1}{2}}} d\lambda = \frac{1}{\tan(\alpha)} \int \frac{\sec^2(\lambda)}{[1 - \tan^2(\lambda)/\tan^2(\alpha)]^{\frac{1}{2}}} d\lambda$$

Substitute $u = \tan(\lambda)/\tan(\alpha)$

Then $d\lambda = \frac{\tan(\alpha)}{\sec^2(\lambda)} du$

$$\begin{aligned} \theta &= \int \frac{1}{[1-u^2]^{\frac{1}{2}}} du \\ &= \sin^{-1}\left(\frac{\tan(\lambda)}{\tan(\alpha)}\right) - \beta \end{aligned}$$

As $\frac{d}{du} \sin^{-1}(u) = \frac{1}{\sqrt{1-u^2}}$

The solution

$$\sin(\theta+\beta) = \frac{\tan(\lambda)}{\tan(\alpha)}$$

Note we can write this

$$\begin{aligned}
\sin(\theta+\beta) &= \frac{1}{\tan(\alpha)}\frac{\sin(\lambda)}{\cos(\lambda)} \\
\tan(\alpha)\cos(\lambda)\sin(\theta+\beta) &= \sin(\lambda) \\
\tan(\alpha)\cos(\lambda)\left[\sin(\theta)\cos(\beta)+\cos(\theta)\sin(\beta)\right] &= \sin(\lambda)
\end{aligned}$$

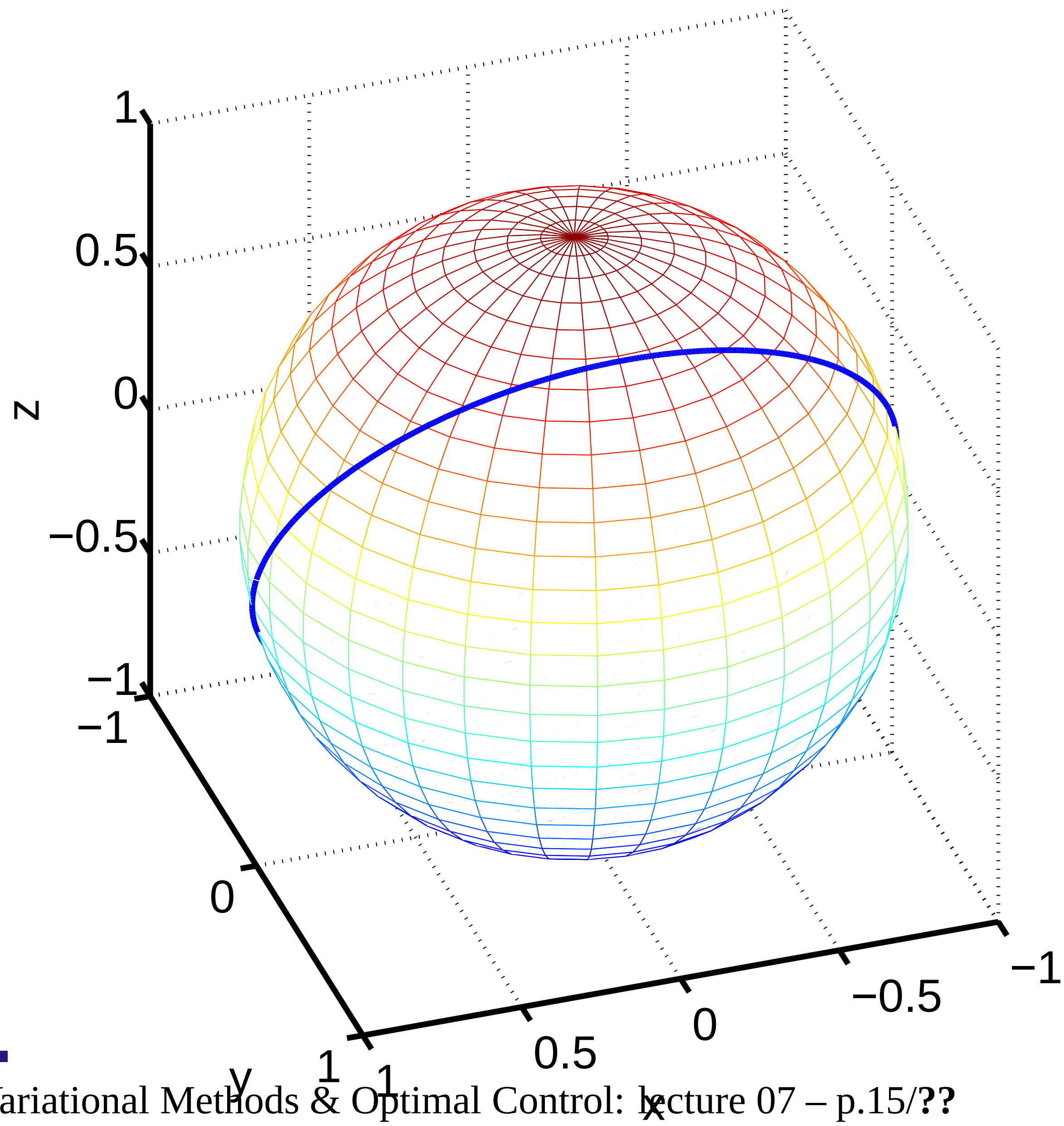
Convert back to Cartesian co-ordinates,

$$\tan(\alpha)\sin(\beta)x+\tan(\alpha)\cos(\beta)y = z$$

which is the equation of a plane, through the origin.
Hence, solution is a **great circle**, the intersection of plane (through the origin) and the sphere.

Example

Ve can find the solution because
ree points (the origin plus the
art and end point of the curve)
efine a plane, and therefore the
olution is the intersection of this
ane with the sphere.



Co-ordinate transformation

More generally, spherical co-ordinates

$$
\begin{aligned}
x &= r\cos(\theta)\cos(\lambda) \\
y &= r\sin(\theta)\cos(\lambda) \\
z &= r\sin(\lambda)
\end{aligned}
$$

And

$$
\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = J \begin{pmatrix} d\theta \\ d\lambda \\ dr \end{pmatrix}, \quad J = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \lambda} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \lambda} & \frac{\partial y}{\partial r} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \lambda} & \frac{\partial z}{\partial r} \end{pmatrix}
$$

Where J is the Jacobian matrix

Jacobians

If

$$\mathbf{y} = \begin{pmatrix} y_1(\mathbf{x}) \\ y_2(\mathbf{x}) \\ \vdots \\ y_n(\mathbf{x}) \end{pmatrix}$$

Then the Jacobian matrix is

$$J(\mathbf{x}) = \frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{pmatrix}$$

The Jacobian determinant

Then the determinant of the Jacobian matrix is also sometimes called the Jacobian

$$|J(\mathbf{x})| = \left|\frac{\partial \mathbf{y}}{\partial \mathbf{x}}\right|$$

This gives the ratios of n-dimensional volumes between the two co-ord. systems, i.e.

$$d\mathbf{y} = |J(\mathbf{x})|\, d\mathbf{x}$$

Transforms and integrals

Substitution in 1D: $y = \phi(x)$

$$\int_{x_0}^{x_1} f(\phi(x))\frac{d\phi}{dx}dx = \int_{\phi(x_0)}^{\phi(x_1)} f(y)dy$$

In 2D

$$\int_R f(x,y)\,dx\,dy = \int_{R^*} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Geodesics

Can we find a geodesic on other surfaces in $\mathbb{R}^3$?

Consider a surface parameterized by $x = x(u,v)$, $y = y(u,v)$, and $z = z(u,v)$, and minimize the arc length

$$L = \int ds = \int \sqrt{dx^2 + dy^2 + dz^2}$$

$$\begin{aligned} dx &= \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \\ dx^2 &= \left(\frac{\partial x}{\partial u}\right)^2 du^2 + 2 \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} du\, dv + \left(\frac{\partial x}{\partial v}\right)^2 dv^2 \end{aligned}$$

and likewise for dy^2 and dz^2.

Geodesics

So we can write the path length as

$$L = \int \sqrt{P + 2Qv' + Rv'^2}\, du$$
$$= \int \sqrt{Pu'^2 + 2Qu' + R}\, dv$$

where $u' = du/dv$ and $v' = dv/du$ and

$$P = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2$$
$$Q = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}$$
$$R = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2$$

Geodesics

Then the Euler-Lagrange equations become

$$\frac{\frac{\partial P}{\partial v} + 2v' \frac{\partial Q}{\partial v} + v'^2 \frac{\partial R}{\partial v}}{2\sqrt{P + 2Qv' + Rv'^2}} - \frac{d}{du}\left(\frac{Q + Rv'}{\sqrt{P + 2Qv' + Rv'^2}}\right) = 0$$

References:

http://mathworld.wolfram.com/GreatCircle.html

http://mathworld.wolfram.com/Geodesic.html