Variational Methods & Optimal Control

lecture 28

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Feedback control systems

In all of our previous examples, we solve optimization problem “all at once”, i.e., we plan the shape of the curve y to optimize the functional. However, sometimes, we need a control that reacts continuously to perturbations in a system. Such controllers typically utilize feedback.

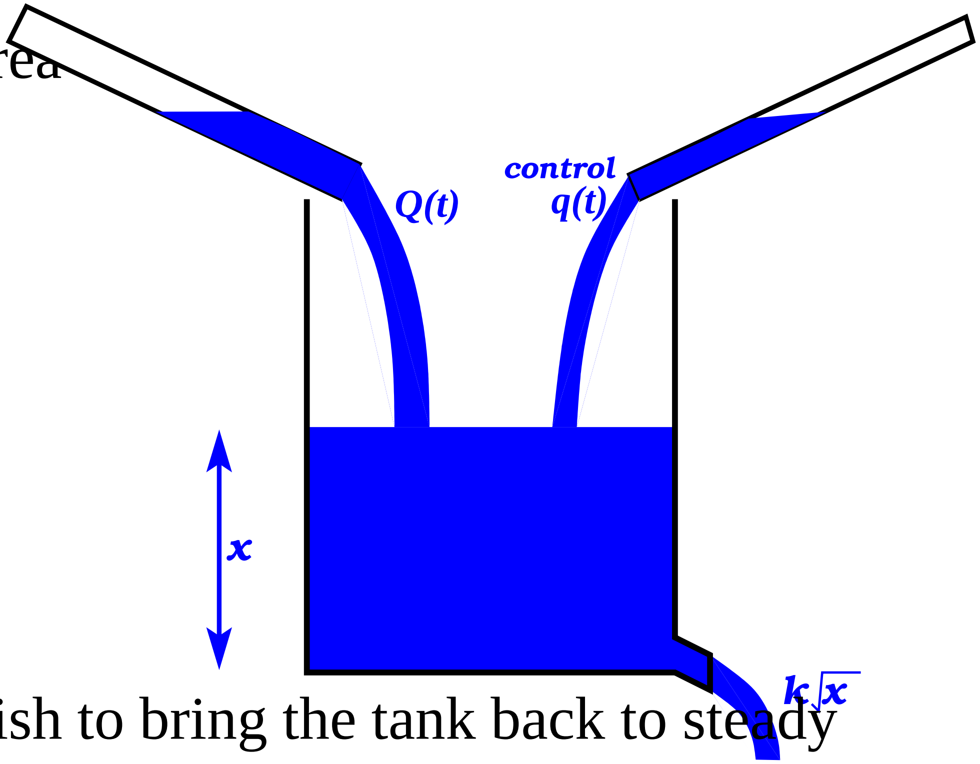
Feedback control systems

- control problem until now have been planned.
  - know the problem before hand
  - assume state is perfectly observable
  - plan the control from the start
- alternative: feedback control
  - observe the state at time t
  - make decisions on the best control at time t
  - continually update this decision

Feedback control example

Liquid level control system

- Tank of uniform cross-sectional area
- Fluid pumped in at rate $Q(t)$
- Fluid level $x(t)$
- Flow out $k\sqrt{x}$
- Given constant flow, then steady-state will be $x_s = (Q/k)^2$



- when there are fluctuations, we wish to bring the tank back to steady state, by addition of suitable input at rate $q(t)$.

Feedback control example

Liquid level optimal control problem

- we wish to operate near steady state, so part of cost is the square deviation

$$\int_0^T [x(t) - x_s]^2 \, dt$$

- Also want to minimize control expenditure, e.g.

$$\int_0^T q(t)^2 \, dt$$

- Problem is to minimize a linear combination of these two, e.g.

$$\int_0^T \left[(x(t) - x_s)^2 + \alpha^2 q(t)^2 \right] \, dt$$

Feedback control example

The dynamics of the system are

$$\dot{x} = Q + q - k\sqrt{x}$$

We can linearize the problem as follows:

- note we wish to maintain x near x_s.
- can approximate x near x_s by a Taylor series

$$\sqrt{x} = \sqrt{x_s} + \frac{1}{2\sqrt{x_s}}(x - x_s) + O\left((x - x_s)^2\right)$$

- the steady state will be when $\dot{x} = 0$, so $Q = k\sqrt{x_s}$
- change variables to $y = x - x_s$, so given $A = k/2\sqrt{x_s}$

$$\dot{y} = -Ay + q(t)$$

Feedback control example

So the optimal control problem can be written as minimize

$$F\{y\} = \int_0^T \left[y(t)^2 + \alpha^2 q(t)^2 \right] \, dt$$

subject to $\dot{y} = -Ay + q(t)$, and $y(0) = x(0) - x_s$ and $y(T) = 0$.

Substitute $q(t) = Ay + \dot{y}$ into the integral and we get

$$F\{y\} = \int_0^T \left[y(t)^2 + \alpha^2 (Ay + \dot{y})^2 \right] \, dt$$

so the E-L equations are

$$\frac{d}{dt}\frac{\partial f}{\partial \dot{y}} - \frac{\partial f}{\partial y} = 2\alpha^2 \frac{d}{dt}\left[\dot{y} + Ay \right] - 2y - 2\alpha^2 \left[A\dot{y} + A^2 y \right] = 0$$

Feedback control example

$$
\begin{aligned}
2\alpha^2 \frac{d}{dt}\left[\dot{y} + Ay\right] - 2y - 2\alpha^2\left[A\dot{y} + A^2 y\right] &= 0 \\
\alpha^2 \frac{d}{dt}\left[\dot{y} + Ay\right] - y - \alpha^2\left[A\dot{y} + A^2 y\right] &= 0 \\
\alpha^2\left[\ddot{y} + A\dot{y} - A\dot{y} - A^2 y\right] - y &= 0 \\
\ddot{y} - \left[\frac{1+\alpha^2 A^2}{\alpha^2}\right] y &= 0
\end{aligned}
$$

This has solution

$$y(t) = Ce^{\lambda t} + Be^{-\lambda t}$$

where $\lambda = \sqrt{\frac{1+\alpha^2 A^2}{\alpha^2}}$.

Feedback control example

The solution is

$$y(t) = Ce^{\lambda t} + Be^{-\lambda t}$$

Given the end-point constraint that $y(0) = y_0$, we get conditions $C + B = y_0$, so the solution is in the form

$$y(t) = y_0 \left(ae^{\lambda t} + (1-a)e^{-\lambda t} \right)$$

Take the case where $T \to \infty$, and we wish $y(T) = 0$, i.e. the fluctuation should go to zero in the far future, then, we require $a = 0$, and the solution will be

$$y(t) = y_0 e^{-\lambda t}$$

where $\lambda = \sqrt{\frac{1+\alpha^2 A^2}{\alpha^2}}$. We get $q(t)$ from $q(t) = Ay + \dot{y}$ which gives

$$q(t) = (A - \lambda) y_0 e^{-\lambda t} = (A - \lambda) y(t)$$

Feedback control example

- the above gives the optimal control policy $q(t)$ over the whole interval $[0, T]$
- actually, a feedback control problem would be more convenient
  - given the state $y(t)$, what should the control be
  - write the control as a function of y, e.g. $q(y)$
- example: **proportional control**

$$q(y) = My$$

where M is called the **gain**

  - we choose proportional control because in previous planned solution $q(t) = (A - \lambda)y_0 e^{-\lambda t}$ which is proportional to the perturbation $y(t)$

Feedback control example

Given a proportional controller, it is easy to rewrite the dynamic equation

$$\dot{y} = -Ay + q(t)$$

using $q(y) = My$ as

$$\dot{y} + (A - M)y = 0$$

which has solution

$$y(t) = y_0 e^{-(A-M)t}$$

We wish to choose M so that it minimizes

$$F\{y\} = \int_0^T \left[y(t)^2 + \alpha^2 q(t)^2 \right] dt$$

Feedback control example

$$
\begin{aligned}
F\{y\} &= \int_0^T \left[y(t)^2 + \alpha^2 q(t)^2\right] dt \\
&= \int_0^T \left[y(t)^2 + \alpha^2 M^2 y(t)^2\right] dt \\
&= \left(1+\alpha^2 M^2\right) \int_0^T y(t)^2 \, dt \\
&= y_0 \left(1+\alpha^2 M^2\right) \int_0^T e^{-2(A-M)t} \, dt \\
&= y_0 \frac{\left(1+\alpha^2 M^2\right)}{-2(A-M)} \left[e^{-2(A-M)t}\right]_0^T
\end{aligned}
$$

Feedback control example

Take the case $T \to \infty$ and we get

$$F\{y\} = y_0 \frac{\left(1+\alpha^2 M^2\right)}{2(A-M)}$$

In order to find a maximum we differentiate WRT M, to get

$$\begin{aligned} \frac{d}{dM} F\{y\} &= y_0 \frac{\alpha^2 2M2(A-M) + 2\left(1+\alpha^2 M^2\right)}{4(A-M)^2} \\ &= y_0 \frac{2\alpha^2 M(A-M) + \left(1+\alpha^2 M^2\right)}{2(A-M)^2} \end{aligned}$$

To get a minimum, the numerator must be zero, so we set

$$2\alpha^2 M(A-M) + \left(1+\alpha^2 M^2\right) = 0$$

Feedback control example

$$\begin{aligned} 2\alpha^2 M(A-M) + \left(1+\alpha^2 M^2\right) &= 0 \\ -\alpha^2 M^2 + 2\alpha^2 AM + 1 &= 0 \\ M^2 - 2AM - 1/\alpha^2 &= 0 \end{aligned}$$

This is just a quadratic equation in M, which we solve to get

$$M = \frac{2A \pm \sqrt{4A^2 + 4/\alpha^2}}{2} = A \pm \sqrt{A^2 + 1/\alpha^2}$$

If we take the solution with the '+' sign, then $F\{y\} = y_0 \frac{\left(1+\alpha^2 M^2\right)}{2(A-M)}$ will be negative (the denominator is negative), so we are restricted to

$$M = A - \sqrt{A^2 + 1/\alpha^2}$$

Feedback control example

- $M < 0$, which makes sense, because the control has to reverse fluctuations.
- Compare the feedback and planned solutions

$$\begin{aligned} q(t) &= (A-\lambda) y_0 e^{-\lambda t} \\ q(t) &= M y(t) = \left(A - \sqrt{A^2 + 1/\alpha^2} \right) y_0 e^{-(A-M)t} \end{aligned}$$

  where $\lambda = \sqrt{\frac{1+\alpha^2 A^2}{\alpha^2}}$, and notice they are the same.
- for α large, e.g. high control cost, the solution has $1/\alpha^2 \to 0$, and so $M \to 0$.
- for α small, e.g. low control cost, the fluctuation cost dominates, and $M \to -1/\alpha$