Tutorial 3: Wednesday 29th August

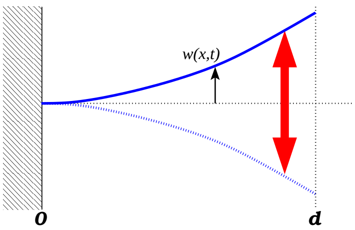
Extensions of the Euler-Lagrange equations and numerical techniques:

1. **Higher-order derivatives:** Go through the steps of deriving the Euler-Poisson equation for a functional containing derivatives or order three, i.e.,

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y', y^{(2)}, y^{(3)}), dx.$$

2. **Multiple dependent variables:** calculate the form of geodesics in N-dimensional Euclidean space.

3. **Multiple independent variables:** take a beam length d with flexural rigidity κ and density per unit length ρ, fixed and clamped at one end, and derive the motion of this beam when one end is held (displaced from its equilibrium position) and then released suddenly (see figure). NB: $\kappa = EI$ where E is the Young's modulus, and I is the area of moment of inertia of the beam.



Hints:

- Assume the beam is thin, and it is not bent too far.
- Ignore gravitational potential for the purpose of solving this problem, and assume deflections are small enough that the beam can be modelled by considering only vertical deflections, so that we can see the notation that the displacement of the beam at distance x from the clamp and time t is $w(x, t)$. We will use w_x and w_t as shorthand for the relevant partial derivatives.
- The boundary conditions for $w(x, t)$ will be

$$\begin{aligned}
w(0,t) &= 0, && \text{because the left end point is fixed} \\
w_x(0,t) &= 0, && \text{because the left end point is clamped} \\
w_t(x,0) &= 0, && \text{because at the start the beam is stationary} \\
w_{xx}(d,t) &= 0, && \text{because the free end point has zero bending moment} \\
w_{xxx}(d,t) &= 0, && \text{because the free end point has zero shearing force}
\end{aligned}$$

where the shape $y(x)$ is determined by the force being applied to the beam before it is released.

- You may assume the solution is separable, i.e., that $w(x,t) = h(x)g(t)$, i.e., that we are looking for a "normal mode" of vibration in which all the components of the beam move with the same frequency and in phase.

4. **Ritz's Method:** Use Ritz's method to find an approximate solution to minimize the

$$J\{y\} = \int_0^{2\pi} y'^2 + \lambda^2 y^2 \, dx,$$

where $y(0) = 1$ and $y(2\pi) = 1$ and λ is a positive integer. Use the trial functions

$$\phi_n(x) = \cos(nx).$$

Compare your solution to one found directly from the Euler-Lagrange equations.