FORWARD DELAY TIMES IN MULTI–PHASE DISCRETE–TIME RENEWAL PROCESSES

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A type of discrete–time Markov renewal process is considered in which the epochs at which the process changes state are not all regeneration points. Such processes arise naturally in breakdown/repair models and in variants of the $M/G/1$ queue such as occur in modelling road traffic. The key renewal theorem is used to find a relationship between the residual sojourn times (or forward delay times) and sojourn times through the system.
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1 Introduction

Renewal processes and Markov renewal processes (MRP) (Pyke, 1961a,b) have been used heavily in stochastic modelling. Nakagawa and Osaki (1976) proposed a new type of renewal process in which not every state transition occurs at a renewal point. Such processes are covered by a general result of Marlow and Tortorella (1995), which considers a reliability process in which the operating and repair times are not necessarily independent. One example proposed by Nakagawa and Osaki (Type 1–MRP) is a modified MRP with N states which are entered sequentially and of which $N-1$ are non–regeneration states. We refer to this process as a multi–phase discrete–time renewal process.

Such processes can be used to model many reliability problems. As a simple example, consider a breakdown/repair model in which the system undergoes a series of breakdowns and repair cycles. The repair and breakdown times are not independent but successive cycles are. Another example is the N–redundant system. In this N units start operating simultaneously. The system continues until all of the units have failed, when they are all replaced and the system begins afresh. The units are considered to operate independently according to a common failure–time distribution and the state is the current number of failed units.

With the aid of technical artifices discussed by Roughan (1994, 1996), these processes may be used also to study types of modified $M/G/1$ queue through the process embedded at departure points. The modification consists of changing the service rate with the next service starting after the queue size has passed upwards or downwards through certain threshold values. This can be used to model many systems with load–dependent service rates. An early applications was to minor road traffic at a T–junction, where a vehicle has different operating characteristics depending on whether it can or cannot proceed immediately on reaching the intersection (see Yeo (1962) and Welch (1964)).

In their study Nakagawa and Osaki considered the non–lattice case and calculated the first–passage times between states and the mean number of entries into a state during a given period of time. With applications in mind, we consider only the lattice or discrete–time case. Many current systems are digital in nature or are observed only at discrete time intervals. Discrete–time analysis is also often preferable even for continuous–time models, as with the modified $M/G/1$ models mentioned above.

The derivation of results equivalent to those of Nakagawa and Osaki for the lattice case is straightforward and will not be addressed here. We concentrate on the relationship between residual sojourn (or forward delay) times and sojourn times through the system. By way of example, consider the behaviour of the breakdown/repair model above when we do not know how long the system has been running. The times until the next breakdown and the succeeding repair time constitute the residual sojourn times. It is useful to be able to calculate the joint probability distribution of this given the probability distribution of the sojourn times, which can be measured directly.

The relationship has more general application, being based on a reformulation of the renewal equation for multi–phase renewal processes. The result may be appplied to any quantity that can be described through a renewal equation. The relationship is established by use of the key renewal theorem of Smith (1958).

Nakagawa and Osaki considered a number of other renewal processes in which the states are not ordered. Such processes may be transformed into a multi–phase renewal process by a transformation procedure described in Roughan (1994).

The paper is organised as follows. Section 2 defines the process, notation and terminology and Section 3 provides the basic results. The main result is the renewal equation of Theorem 1. This is used in Theorem 2 to calculate the joint probability generating function (PGF) of the residual sojourn times in terms of that of the sojourn times. Finally, in Section 4, a simple illustrative example is given.

2 Definitions

Consider a Markov renewal process which passes cyclically through the states 0, 1, ... , $N-1$. State 0, which it will be convenient notationally to also label as state N in some specific situations, is a regeneration state. This is the Type 1–MRP of Nakagawa and Osaki (1976). We make the aperiodicity assumption that the probability distribution of the first–passage times is lattice with period 1.

Define $\tau_l^m \in \mathbb{Z}^+$ to be the epoch of the mth entry into state l. We call the period $[\tau_0^m, \tau_N^m)$ the mth cycle. From this we can define the state sojourn lifetimes i_j of the mth cycle by

$$i_j = \tau_j^m - \tau_{j-1}^m \qquad \text{for } 1 \le j \le N, \quad m \ge 1.$$

Here we identify the mth entry into state N with the $(m+1)$th entry into state 0, that is, $\tau_N^m = \tau_0^{m+1}$.

We allow interdependence between state sojourn lifetimes within a cycle but assume independence between cycles. This is essential to the idea that the entry epochs of the states 1 to $N-1$ are non–regeneration points, while the entry epochs of state 0 are regeneration points. We denote by $f(i_1, i_2, \ldots, i_N)$ the joint probability distribution of the state sojourn lifetimes and take $i_1 + i_2 + \cdots + i_N \ge 1$ with probability 1. Define $\rho = E(i_1 + \cdots + i_N)$.

If the process is in the mth cycle and state k at time n, the residual sojourn time r_j for state j is defined as

$$r_j = \begin{cases} 0 & j \le k \\ \tau_j^m - n & j = k+1 \\ \tau_j^m - \tau_{j-1}^m & j > k+1 \end{cases}$$

for $1 \le j \le N$. Note that for $j > k+1$ the residual sojourn time is just the state sojourn lifetime, that is, $i_j = r_j$.

To obviate overly cumbersome expressions for multiple sums, we shall employ the notation

$$\sum_{\spadesuit l} = \sum_{i_{l+1}=0}^{\infty} \sum_{i_{l+2}=0}^{\infty} \cdots \sum_{i_N=0}^{\infty},$$
$$\sum_{\heartsuit l} = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_l=0}^{\infty}.$$

We are now in a position to address the probability distributions

$$\begin{aligned} q^n(r_1, r_2, .., r_N) &= \text{prob. at time } n \text{ that the residual sojourn time for} \\ &\quad \text{state } j \text{ is } r_j \ (j = 1, \ldots, N\ ), \\ p^{nl}(r_{l+1}, r_{l+2}, .., r_N) &= \text{prob. at time } n \text{ that the process is in state } l \text{ and that the} \\ &\quad \text{residual sojourn time for state } j \text{ is } r_j \ (j = l+1, \ldots, N\ ), \\ q(r_1, r_2, .., r_N) &= \lim_{n\to\infty} q^n(r_1, .., r_N) \text{ (when this exists)}, \\ p^l(r_{l+1}, r_{l+2}.., r_N) &= \lim_{n\to\infty} p^{nl}(r_{l+1}, .., r_N) \text{ (where this exists)}, \qquad (1) \\ f^l(i_{l+1}, .., i_N) &= \sum_{\heartsuit l} f(i_1, i_2, .., i_N) \qquad (2) \\ &= \text{joint probability distribution for the times } (i_{l+1}, \ldots, i_N) \end{aligned}$$

and define corresponding PGFs

$$\begin{aligned} Q^*(x_1, x_2, \ldots, x_N) &= \sum_{\spadesuit 0} \left(\prod_{k=1}^{N} x_k^{r_k} \right) q(r_1, r_2, .., r_N), \\ P_l^*(x_{l+1}, x_{l+2}, \ldots, x_N) &= \sum_{\spadesuit l} \left(\prod_{k=l+1}^{N} x_k^{r_k} \right) p^l(r_{l+1}, .., r_N), \\ F^*(x_1, x_2, \ldots, x_N) &= \sum_{\spadesuit 0} \left(\prod_{k=1}^{N} x_k^{i_k} \right) f(i_1, i_2, .., i_N), \\ F_l^*(x_{l+1}, x_{l+2}, \ldots, x_N) &= \sum_{\spadesuit l} \left(\prod_{k=l+1}^{N} x_k^{i_k} \right) f^l(i_{l+1}, i_{l+2}, .., i_N), \\ F_N^* &= 1 \end{aligned}$$

for $x_1, x_2, \ldots, x_N \in [0, 1]$. Thus Q^* is the PGF for the residual sojourn times and F^* that of the state sojourn lifetimes.

Note that $F_l^*(x_{l+1}, x_{l+2}, \ldots, x_N) = F^*(1, \ldots, 1, x_{l+1}, \ldots, x_N)$. Also, from the theorem of total probability, we get

$$q^n(r_1, r_2, .., r_N) = \sum_{l=0}^{N-1} p^{nl}(r_{l+1}, r_{l+2}, .., r_N),$$

so that

$$Q^*(x_1, x_2, \ldots, x_N) = \sum_{l=0}^{N-1} P_l^*(x_{l+1}, x_{l+2}, \ldots, x_N). \tag{3}$$

Finally we define

$$h^l(n) \quad = \quad \text{prob. of state } l \text{ being entered at time } n.$$

3 Results

Our results are encapsulated in the following two theorems.

Theorem 1 *The renewal equation is*

$$p^{n0}(r_1, \ldots, r_N) = f(r_1 + n, \ldots, r_N) + \sum_{m=1}^{n-1} h^0(n-m) f(r_1 + m, \ldots, r_N). \tag{4}$$

If $\rho < \infty$, then for $0 \le l < N$

$$h^l(n) \to \frac{1}{\rho}$$

and

$$p^{nl}(r_{l+1}, \ldots, r_N) \to \frac{1}{\rho} \sum_{m=1}^{\infty} f^l(r_{l+1} + m, \ldots, r_N) \tag{5}$$

as $n \to \infty$.

Proof. The epochs of entry into state 0 form a renewal process with probability distribution given by $f(i) = \sum_{i_1 + \cdots + i_N = i} f(i_1, \ldots, i_N)$. Aperiodicity hs been assumed, so as $\rho < \infty$ we have from elementary renewal theory (Wolff, 1989, Thm 18, p. 116) that

$$h^0(n) \to \frac{1}{\rho}.$$

Since each state is entered exactly once in each cycle,

$$h^l(n) - h^0(n) \to 0, \qquad \forall l: \; 1 \le l < N,$$

whence we have the first part of (5).

The renewal equation comes directly from standard renewal arguments. Application of the lattice version of the key renewal theorem (Wolff, 1989, Thm 19, p. 117) to (4) provides the second part of (5) for $l = 0$. For $l > 0$ we argue as follows. When the current state is l, the forward recurrence times are given by

$$(r_1, .., r_l, r_{l+1}, r_{l+2}, .., r_N) = (0, .., 0, i_{l+1} + m, i_{l+2}, .., i_N).$$

To obtain $p^{nl}(r_{l+1},\ldots,r_N)$, we must sum over all possible values of $i_1,...,i_l$ and m, multiplied by $h^l(n-m)$. Since $h^l(n-m)\to\frac{1}{\rho}$ as $n\to\infty$, we have

$$p^{nl}(r_{l+1},\ldots,r_N) \quad\to\quad \sum_{\heartsuit l}\frac{1}{\rho}\sum_{m=1}^{\infty} f(i_1,...,i_l,r_{l+1}+m,\ldots,r_N),$$

which provides the result. □

Theorem 2 *If* $x_1,x_2,\ldots,x_N\in[0,1]$*, then*

$$Q^*(x_1,x_2,\ldots,x_N)=\frac{1}{\rho}\sum_{l=1}^{N}\frac{F_l^*(x_{l+1},\ldots,x_N)-F_{l-1}^*(x_l,x_{l+1},\ldots,x_N)}{1-x_l}. \tag{6}$$

Furthermore, if the series definition of Q^* *converges for some* $x_1,x_2,\ldots,x_N$ *not all in the interval* $[0,1]$*, then it converges to the right–hand side of (6).*

Proof. By Theorem 1 and definitions (1) and (2), we have that

$$p^{l-1}(i_l,i_{l+1},..,i_N) \quad=\quad \frac{1}{\rho}\left\{f^l(i_{l+1},\ldots,i_N)-\sum_{m=0}^{i_l}f^{l-1}(m,..,i_N)\right\}.$$

Multiplication of the left–hand side by $\prod_{k=l}^{N}x^{i_k}$ and summation over i_k for $k=l,\ldots,N$ provides (for $l>0$) the generating function P_{l-1}^*. Performing the same operations on the right–hand side and exchanging the summations over i_l and m yields

$$\begin{aligned}
P_{l-1}^*(x_l,x_{l+1},\ldots,x_N) &= \frac{1}{\rho}\left\{\sum_{\spadesuit l}\left(\prod_{k=l+1}^{N}x_k^{i_k}\right)f^l(i_{l+1},\ldots,i_N)\left(\frac{1}{1-x_l}\right)\right.\\
&\qquad\left.-\sum_{\spadesuit l}\left(\prod_{k=l+1}^{N}x_k^{i_k}\right)\sum_{m=0}^{\infty}f^{l-1}(m,i_{l+1},..,i_N)\left(\frac{x_l^m}{1-x_l}\right)\right\}\\
&= \frac{1}{\rho}\frac{1}{1-x_l}\left\{\sum_{\spadesuit l}\left(\prod_{k=l+1}^{N}x_k^{i_k}\right)f^l(i_{l+1},\ldots,i_N)\right.\\
&\qquad\left.-\sum_{\spadesuit l-1}\left(\prod_{k=l}^{N}x_k^{i_k}\right)f^{l-1}(i_l,..,i_N)\right\}\\
&= \frac{1}{\rho}\left\{\frac{F_l^*(x_{l+1},\ldots,x_N)-F_{l-1}^*(x_l,x_{l+1},\ldots,x_N)}{1-x_l}\right\}. \qquad (7)
\end{aligned}$$

Relation (3) now provides (6).

Now if Q^* converges for some $x_j\in(1,\alpha]$, we must show that it converges to the right–hand side of (6). Clearly if x_i lies in $[0,1)$, then (7) holds for $l=i$. For $l=j$ we proceed as follows.

$$P_{l-1}^*(x_l,x_{l+1},\ldots,x_N) \quad=\quad \frac{1}{\rho}\left\{\sum_{\spadesuit l-1}\left(\prod_{k=l}^{N}x_k^{i_k}\right)\sum_{m=1}^{\infty}f^{l-1}(i_l+m,\ldots,i_N)\right\}$$

$$
\begin{aligned}
&= \frac{1}{\rho}\left\{\sum_{\spadesuit l}\left(\prod_{k=l+1}^{N} x_k^{i_k}\right)\sum_{m=1}^{\infty} x_l^{-m}\left[\sum_{i_l=0}^{\infty} x_l^{i_l} f^{l-1}(i_l,\ldots,i_N)\right.\right.\\
&\qquad\left.\left. - \sum_{i_l=0}^{m-1} x_l^{i_l} f^{l-1}(i_l,\ldots,i_N)\right]\right\}\\
&= \frac{1}{\rho}\left\{\sum_{\spadesuit l}\left(\prod_{k=l+1}^{N} x_k^{i_k}\right)\sum_{i_l=0}^{\infty} x_l^{i_l} f^{l-1}(i_l,\ldots,i_N)\sum_{m=1}^{\infty} x_l^{-m}\right.\\
&\qquad\left. - \sum_{\spadesuit l}\left(\prod_{k=l+1}^{N} x_k^{i_k}\right)\sum_{i_l=0}^{\infty}\sum_{m=i_l+1}^{\infty} x_l^{i_l-m} f^{l-1}(i_l,\ldots,i_N)\right\}\\
&= \frac{1}{\rho}\sum_{m=1}^{\infty} x_l^{-m}\left\{\sum_{\spadesuit l-1}\left(\prod_{k=l}^{N} x_k^{i_k}\right) f^{l-1}(i_l,\ldots,i_N)\right.\\
&\qquad\left. - \sum_{\spadesuit l}\left(\prod_{k=l+1}^{N} x_k^{i_k}\right)\sum_{i_l=0}^{\infty} f^{l-1}(i_l,\ldots,i_N)\right\}\\
&= \frac{1}{\rho}\frac{x_l^{-1}}{1-x_l^{-1}}\left\{F_{l-1}^*(x_l,x_{l+1},\ldots,x_N) - F_l^*(x_{l+1},\ldots,x_N)\right\}\\
&= \frac{1}{\rho}\left\{\frac{F_l^*(x_{l+1},\ldots,x_N) - F_{l-1}^*(x_l,x_{l+1},\ldots,x_N)}{1-x_l}\right\},
\end{aligned}
$$

that is, (7) holds. Thus we see that (6) still applies. □

Remark: Since $F_l^*(x_{l+1},x_{l+2},\ldots,x_N) = F^*(1,\ldots,1,x_{l+1},\ldots,x_N)$, the relationship in Theorem 2 is between the PGF of the residual sojourn times and that of the state sojourn lifetimes.

4 An example

To conclude, we present a simple illustrative example. Consider a breakdown/repair model in which the time to the first breakdown is geometrically distributed and the repair time is always equal to the time spent running the system before the breakdown, that is, $f^0(i_1,i_2) = (1-p)p^{i_1-1}\delta_{i_1,i_2}$ for $i_1,i_2 = 1,2,\ldots$, where $\delta_{i,j}$ is the Kronecker delta. Then

$$
\begin{aligned}
F_0^*(x_1,x_2) &= \frac{x_1x_2(1-p)}{1-px_1x_2},\\
\rho &= 2/(1-p),\\
Q^*(x_1,x_2) &= \frac{1}{\rho}\left[\frac{F_1^*(x_2)-F_0^*(x_1,x_2)}{1-x_1} + \frac{1-F_1^*(x_2)}{1-x_2}\right]\\
&= \frac{1-p}{2}\left[\frac{\frac{x_2(1-p)}{1-x_2p} - \frac{x_1x_2(1-p)}{1-x_1x_2p}}{1-x_1} + \frac{1-\frac{x_2(1-p)}{1-x_2p}}{1-x_2}\right]
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1-p}{2}\left[\frac{x_2(1-p)(1-x_1x_2p) - x_1x_2(1-p)(1-x_2p)}{(1-x_1)(1-x_2p)(1-x_1x_2p)} + \frac{1-x_2p-x_2(1-p)}{(1-x_2)(1-x_2p)}\right] \\
&= \frac{1-p}{2}\left[\frac{x_2(1-p)[(1-x_1x_2p)-x_1(1-x_2p)]}{(1-x_1)(1-x_2p)(1-x_1x_2p)} + \frac{1-x_2p-x_2+x_2p)}{(1-x_2)(1-x_2p)}\right] \\
&= \frac{1-p}{2}\left[\frac{x_2(1-p)[1-x_1x_2p-x_1+x_1x_2p]}{(1-x_1)(1-x_2p)(1-x_1x_2p)} + \frac{1-x_2}{(1-x_2)(1-x_2p)}\right] \\
&= \frac{1-p}{2}\left[\frac{x_2(1-p)}{(1-x_2p)(1-x_1x_2p)} + \frac{1}{(1-x_2p)}\right].
\end{aligned}$$

The means may be calculated as

$$\begin{aligned}
E[r_1] &= \left.\frac{\partial}{\partial x_1}Q^*(x_1,x_2)\right|_{x_1=x_2=1} \\
&= \frac{1}{2}\frac{p}{(1-p)} \\
&= P\{phase1\}E[r_1|phase1] + P\{phase2\}E[r_1|phase2], \\
E[r_2] &= \left.\frac{\partial}{\partial x_2}Q^*(x_1,x_2)\right|_{x_1=x_2=1} \\
&= \frac{1-p}{2}\left[\frac{(1-p)^3+2p(1-p)^2}{(1-p)^4} + \frac{p}{(1-p)^2}\right] \\
&= \frac{1}{2}\frac{1}{1-p} + \frac{1}{2}\frac{p}{1-p} \\
&= P\{phase1\}E[r_2|phase1] + P\{phase2\}E[r_2|phase2].
\end{aligned}$$

The last lines in both the above are as would be expected, and indeed this simple example could be analysed by elementary probabilistic arguments. Thus by symmetry, the probabilities of arriving at the system when it in the working and repairing phases must be equal, so that

$$P\{phase1\} = P\{phase2\} = \frac{1}{2}.$$

Similarly $E[r_1|phase2]$ is zero, as once the system is in the repair phase, the residual sojourn time in phase 1 is zero.

The utility of the result appears, of course, with more complex examples. For instance, if in the above example the time until breakdown were generally distributed, so that $f^0(i_1,i_2) = f(i_1)\delta_{i_1,i_2}$ for $i_1, i_2 = 1, 2, \ldots$, then Q^* could be written as

$$Q^*(x_1,x_2) = \frac{1}{\rho}\left[\frac{F^*(x_2)-F^*(x_1x_2)}{1-x_1} + \frac{1-F^*(x_2)}{1-x_2}\right],$$

where $F^*(x_2)$ is the PGF of $f(i)$.

The model can equally be applied to the case when the breakdown moments are renewal times and the start of operation is a non–renewal point. This describes a system where a repair is performed by replacing one component of a system rather than the whole system. The time until the next breakdown will thus depend on which component was replaced.

Another use for the result can be found in Roughan (1996), where Theorem 2 is used to obtain the equilibrium probability density function of the number of customers in a modified $M/G/1$–type system.

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